ESSAYS

501322

ON SEVERAL
Curious and Uſeful SUBJECTS,
In SPECULATIVE and MIX'D

MATHEMATICKS.

Illuſtrated by a Variety of EXAMPLES.

By *THOMAS SIMPSON.*

James Fraser

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G R
BRITISH MUSEUM

TO

FRANCIS BLAKE,

OF

Twiſel, in the County of *Durham*, Eſq;

SIR,

AS our private Correſpondence occaſioned my Drawing up ſeveral of the following Papers, I thence claim a Sort of a Right to addreſs them to You: But I well know the common Style of a DEDICATION would to You be highly offenſive; therefore all the Uſe I dare make of this Opportunity, is, to declare Myſelf to be,

SIR,

Your moſt Obliged,

Humble Servant,

THO. SIMPSON.

PREFACE.

THE Reader, I preſume, will excuſe me, if, inſtead of acquainting him, in the uſual Way, with the many weighty Reaſons that induced me to publiſh the following Sheets, I ſhall take up no more of his Time than to give a conciſe Account of the Nature and Uſefulneſs of the ſeveral Papers that compoſe this Miſcellany, *in the Order they are printed.*

The firſt, then, is concerned in determining the Apparent Place of the Stars ariſing from the progreſſive Motion of Light, and of the Earth in its Orbit; which, though it be a Matter of great Importance in Aſtronomy, and allowed one of the fineſt Diſcoveries, yet had it not been fully and demonſtratively treated of by any Author, or indeed thrown into any Method of Practice. Now, however, I muſt not omit to acknowledge, that in the laſt Volume of the Memoirs *of the* ROYAL ACADEMY *of* SCIENCES, *for the Year* 1737.

lately publiſhed at Paris, *and brought hither a few Weeks ſince, there is a Paper on this Subject by Monſieur* Clairaut, *a very eminent Mathematician of that* Academy; *to which he ſubjoins a Set of Practical Rules for the* Aberration *in* Right-Aſcenſion *and* Declination *only; wherein moſt of his Analogies are exactly the ſame as thoſe inſerted in this Book, with which Dr.* Bevis *favoured me: For which Reaſon I think it proper to aſſure my Readers, that my Paper, together with the Doctor's Rules, were quite printed off, and in the Hands of ſeveral Friends, who deſired them, before* Chriſtmas 1739. *when the Severity of the Seaſon interrupted for a conſiderable Time the Impreſſion of this Treatiſe.*

The ſecond Paper, treats of the Motion of Bodies affected by Projectile and Centripetal Forces; wherein the Invention of Orbits *and the Motion of Apſides, with many others of the moſt conſiderable Matters in the Firſt Book of Sir* Iſaac Newton's PRINCIPIA, *are fully and clearly inveſtigated.*

The Third, ſhews how, from the Mean Anomaly of a Planet given, to find its true Place in its Orbit, by three ſeveral Methods; but what may beſt recommend this Paper, is the Practical Rule in the annexed Scholium, which will, I hope, be found of Service.

The Fourth, includes the Motion and Paths of Projectiles in reſiſting Mediums, in which not only the Equation of the Curve deſcribed according to any Law of Denſity, Reſiſtance, &c. but all the moſt important Matters, upon this Head, in the Second Book of the above-named illuſtrious Author, are determined in a new, eaſy, and comprehenſive Manner.

The Fifth, conſiders the Reſiſtances, Velocities, and Times of Vibration, of pendulous Bodies in Mediums.

The Sixth, contains a new Method for the Solution of all Kinds of Algebraical Equations in Numbers; which, as it is more general than any hitherto given, cannot but be of conſiderable Uſe, though it perhaps may be objected, that the Method of Fluxions, whereon it is founded, being a more exalted Branch of the Mathematicks, cannot be ſo properly applied to what belongs to common Algebra.

The Seventh, relates to the Method of Increments; which is illuſtrated by ſome familiar and uſeful Examples.

The Eighth, is a ſhort Inveſtigation of a Theorem for finding the Sum of a Series of Quantities by Means of their Differences.

The Ninth, exhibits an eaſy and general Way of Inveſtigating the Sum of a recurring Series.

Theſe three laſt Papers relate chiefly to the Inventions of Others: As they are all of Importance, and are required in other Parts of the Book, I could not well leave them entirely untouch'd; and if I ſhall be thought to have thrown any new Light upon them, that may benefit young Proficients, I have my End.

The Tenth, comprehends a new and general Method for finding the Sum of any Series of Powers whoſe Roots are in Arithmetical Progreſſion, which may be applied with equal Advantage to Series of other Kinds.

The Eleventh, is concerned about Angular Sections and ſome remarkable Properties of the Circle.

The Twelfth, includes an eaſy and expeditious Method of Reducing a Compound Fraction to Simple Ones; the firſt Hints whereof I freely acknowledge to have received from Mr. Muller's *ingenious Treatiſe on Conic Sections and Fluxions.*

The Thirteenth and laſt, containing a general Quadrature of Hyperbolical Curves, is a Problem remarkable enough, as well on account of its Difficulty, as its having exerciſed the Skill of ſeveral great Mathematicians; but as none of the Solutions hitherto publiſhed, tho' ſome of them are very elegant ones, extend farther than to particular Caſes, except that given in Phil. Tranſ. N°. 417. *without Demonſtration, I flatter myſelf that this which I have now offered, may claim an Acceptance, ſince it is clearly inveſtigated by two different Methods, without referring to what hath been done by Others, and the general Conſtruction rendered abundantly more ſimple and fit for Practice than it there is.*

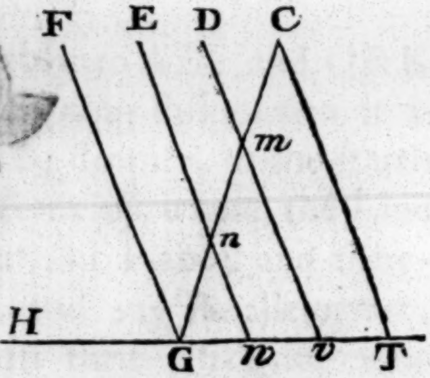
ESSAYS

On ſeveral Curious and Uſeful Subjects in Speculative and Mixt Mathematicks.

Of the Apparent Places of the FIXED STARS, ariſing from the *Motion of Light*, and the *Motion of the Earth in its Orbit*.

PROPOSITION I.

If the Velocity of the Earth in its Orbit bears any ſenſible Proportion to the Velocity of Light, every Star in the Heavens muſt appear diſtant from its true Place; and that by ſo much the more, as the Ratio of thoſe Velocities approaches nearer to that of Equality.

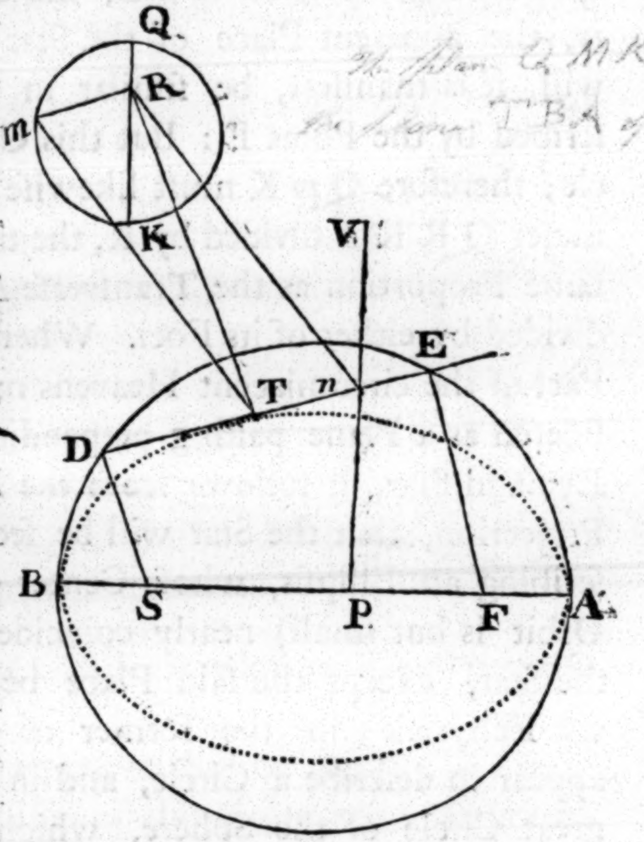


FOR, if while the Line CG is deſcribed by a Particle of Light coming from a Star in that Direction, the Eye of an Obſerver at T be carry'd, by the Earth's Motion, thro' TG; and CT be a Tube made uſe of in obſerving; and a Particle of Light, from the ſaid Star, be juſt entering at C the End of its Axis; then when the Eye is arrived at v, the Tube will have acquired the Poſition vD parallel to TC, and the ſaid Particle will be at the Point m, where the Line CG interſects the Axis of the Tube; becauſe $GT : GC :: Tv : Cm$. Let now the Tube, by the Earth's Motion, be brought into the Poſition Ew; then becauſe $GT : GC :: Tw : Cn$, the Particle will be at n, and therefore is ſtill in the Axis of the Tube: Therefore when it enters the Eye at G, as it has all the Time been in the Axis of the Tube, it muſt conſequently appear to have come in the Direction thereof, or to make an Angle with TH, the Line that the Earth moves in, equal to CTH, which is different from what it really does, by the Angle GCT: Whence it is evident that, unleſs the Earth always moves in a Right Line directly to or from a given Star (which is abſurd to ſuppoſe) that Star muſt appear diſtant from its true Place; and the more ſo, as the Velocity of the Earth (in reſpect of that of Light) is increaſed. And the ſame muſt neceſſarily be the Caſe when the Obſervation is made by the naked Eye; for the Suppoſition and Uſe of a Tube neither alters the real nor apparent Place of the Star, but only helps to a more eaſy Demonſtration.

* * *

PROPOSITION II.

To find the Path which a Star, thro' the aforeſaid Cauſe, in one entire Annual Revolution of the Earth, appears to deſcribe.



The plane QMK [illegible]
the plane TBA of the Ecliptic

LET ATBA be the Orbit of the Earth; S the Sun in one Focus; F the other Focus; T the Earth moving in its Orbit from A towards B; DT*n* a Tangent at T; and SD, FE Perpendiculars thereto: Let Q*m*KRQ be Part of an indefinite Plane parallel to that of the Ecliptick, paſſing thro' R the Centre of the given Star; and take T*n* to TR, as the Velocity of the Earth in its Orbit at T, to that of a Particle of Light coming from the ſaid Star: Let T*m* be parallel to *n*R; P*n*V perpendicular to AB; and QRK parallel to P*n*V: Then from the foregoing Propoſition it is manifeſt, that a Ray of Light coming from R to the Earth at T, will appear as if it proceeded from *m*, where the Line T *m*, produced, interſects the ſaid parallel Plane; and therefore, becauſe T *m* is parallel to R*n*, and any Parallelogram, interſecting two parallel Planes, cuts them alike in every reſpect, it is evident that R *m* muſt be equal to T *n*, and QR *m* to V*n*D; wherefore ſince D and P are equal to

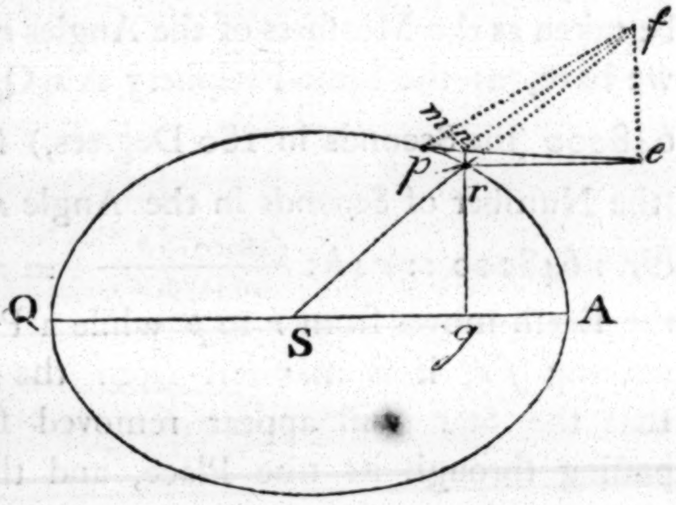
$\angle D = \angle P = 90^\circ$

two

two Right Angles, DSP and DnP muſt be equal, alſo, to two Right Angles, and conſequently QRm (= VnD) = DSP = AFE. But Tn or Rm, expreſſing the Celerity of the Earth at T, is known to be inverſely as SD; or, becauſe $SD \times FE$ is every where the ſame, directly as FE; wherefore the Angles AFE, QRm being every where equal, and Rm in a conſtant Proportion to FE, the Curve QmK deſcribed by m, the apparent Place of the Star in the ſaid parallel Plane, will, it is manifeſt, be ſimilar in all Reſpects to AEB deſcribed by the Point E: But this Curve is known to be a Circle; therefore QmK muſt likewiſe be a Circle, whoſe Diameter QRK is divided by R, the true Place of the Star, in the ſame Proportion as the Tranſverſe Axis of the Earth's Orbit is divided by either of its Foci. Wherefore, foraſmuch as a ſmall Part of the circumjacent Heavens may, in this Caſe, be conſidered as a Plane paſſing perpendicular to a Line joining the Eye and Star, it follows from the Principles of Orthographic Projection, that the Star will be ſeen in the Heavens as deſcribing an Ellipſis, whoſe Center (as the Excentricity of the Orbit is but ſmall) nearly coincides with the true Place of the Star, except the ſaid Place be in the Pole or Plane of the Ecliptick; in the former of which Caſes the Star will appear to deſcribe a Circle, and in the latter an Arch of a great Circle of the Sphere, which by Reaſon of its Smallneſs may be conſidered as a Right Line. But theſe Concluſions will perhaps appear more plain from the next Propoſition, where for the Sake of Eaſe and Brevity, the Earth is conſidered as moving in an Orbit perfectly circular, from which her real Orbit does not greatly differ.

PROPOSITION III.

Having given, from Experiment, the Ratio of the Velocity of Light to that of the Earth in its Orbit, and the true Places of the Sun and a Star; to find the apparent Place of the Star from thence ariſing.



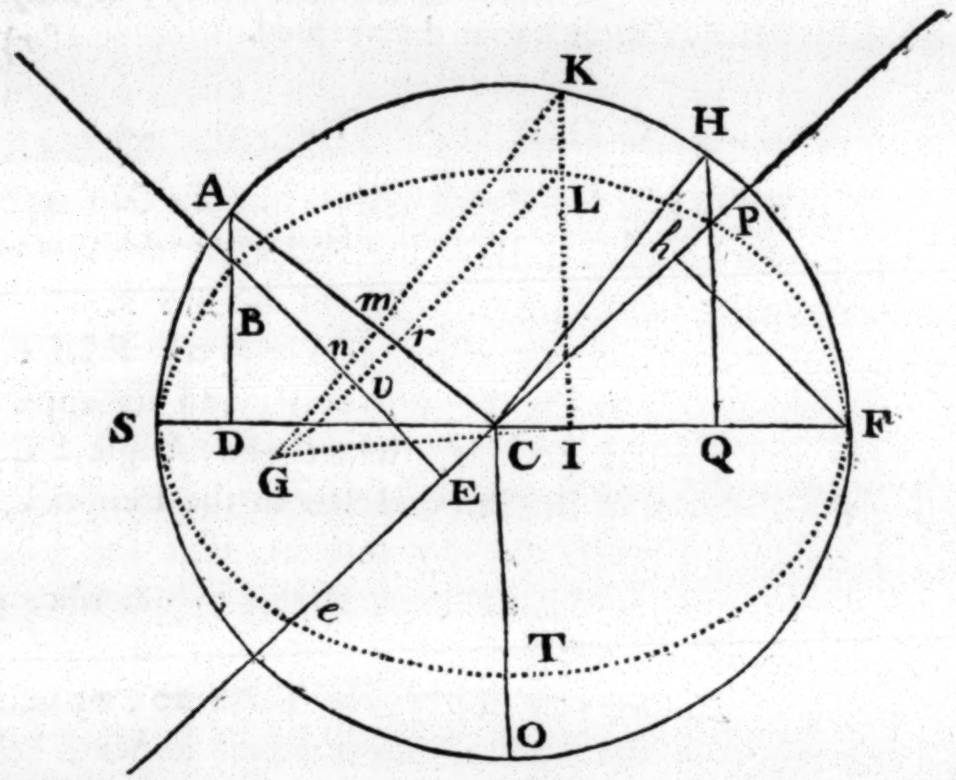
LET A r Q A be the Earth's Orbit, conſidered as a Circle; S the Sun in the Center thereof; r the Earth moving about the ſame from A towards Q; re a Line, which being produced, ſhall paſs thro' the Ecliptick Place of the given Star; A S parallel, and ~~qr~~ perpendicular, thereto: Let ef be perpendicular to the Plane of the Ecliptick, ſo that rf being equal to Sr or Radius, re may be the Coſine of the Latitude of the given Star: This being premiſed, it is manifeſt that the true Place of the Star, from the Earth, will be in the Direction rf, and with Reſpect to the Ecliptick, in the Line re; therefore the Angle Sre ($=$ QSr) being the Difference of Longitudes of the Sun and Star, is given by the Queſtion. Let rg, the Sine of the Supplement of this Angle, be denoted by b; its Coſine Sg, by c; the Sine of the given Latitude, or fe, by s; and the Radius Sr, or fr, by Unity; and while a Particle of Light is moving along fr, let the Earth be ſuppoſed to be carry'd in

$g\,r$

C its

its Orbit from r to p, over a Diſtance ſignified by r; and, pe, pf being drawn, make rn and nm perpendicular thereto: Then becauſe of the exceeding Smallneſs of pr it may be conſidered as a Right-Line; and we ſhall have 1 $(Sr):b$ $(rg)::r$ $(pr):rb(=pn)$; and $1:r::c:rc(=rn)$ (by the Similarity of the Triangles prn, Srg); whence as 1 $(fp,)$ to s (fe) ſo is rb, to $rbs=(nm)$ the Sine of the Angle nfm: But ſince the Sine or Tangent of a very ſmall Arch differs inſenſibly from the Arch itſelf, theſe Values rc and rbs may be taken as the Meaſures of the Angles rfn, and nfm: Hence we have, as the Semi-Periphery ArQ $(=3.14159$, &c.) to 648000 (the Seconds in 180 Degrees,) ſo is rc to $\frac{648000\,rc}{3.14159,\,\&c.}$ (the Number of Seconds in the Angle rfn;) and as 3.14159 &c.$:648000::rsb:\frac{648000\,rsb}{3.14159,\,\&c.}=nfm$: Therefore, as the Earth moves from r to p while a Particle of Light is deſcribing fr, it is manifeſt from the aforeſaid Propoſition, that the Star will appear removed from the great Circle paſſing through its true Place, and the Pole of the Ecliptick by $\frac{648000\,rc}{3.14159,\,\&c.}$ Seconds; and to have its Latitude increaſed by $\frac{648000\,rsb}{3.14159,\,\&c.}$ Seconds. *Q. E. I.*

COROL. I.



HENCE if C be the true Place of the Star, S C F its Parallel of Latitude; and about C, as a Centre, the Ellipſis F P S T F, and Circle F H S O F be deſcribed ſo, that F C may be $= \frac{648000\,r}{3.14159, \&c.}$ and T C, the Semi-Conjugate Axis, in proportion thereto, as s to 1; and if the Angle SCH be taken equal to the Difference of Longitudes of the Sun and Star; then in the Point P, where the elliptical Periphery is interſected by the Right Line H Q falling perpendicularly on FS, the Star will appear to be poſited. For as 1 (Radius) : b (Sine of Q C H) : : C H : $b \times$ C H = H Q; but by the Relation of the two Curves, C H : C T : : $b \times$ C H (= H Q) : P Q, that is, by Conſtruction, $1 : s :: \frac{648000\,r}{3.14159, \&c.} \times b : \frac{648000\,r\,s\,b}{3.14159, \&c.}$ $=$ P Q;

$=PQ$; again as 1 (Radius) : c (the Coſine of QCH) :: $\frac{648000\,r}{3.14159, \&c.}$ ($=CH$) : $\frac{648000\,rc}{3.14159, \&c.}$ $=CQ$; which Expreſſions are the very ſame as thoſe above determined.

COROL. II.

THEREFORE it follows, that while the Sun appears to purſue his Courſe thro' the Ecliptick, the Star will be ſeen as moving from F towards L and S, and ſo on, 'till it hath deſcribed the whole elliptical Periphery FLSTF; that its Latitude will be the the leaſt at T; and its apparent Longitude the greateſt poſſible, when the Angle SCH, ſhewing the Diſtance of the Sun and Star in the Ecliptick, is equal to two right ones. It alſo follows, that the greater Axis of the Ellipſes, which all Stars appear to deſcribe, are equal, and found by Obſervation to amount to 40 ½ Seconds of a great Circle, very nearly; the Term 20″, 25 which frequently occurs in the practical Rules hereto annext, being put for the half thereof. It follows moreover, that the greateſt Aberrations, or *Maxima*, in Longitude, will be as the Coſines of the Latitudes inverſely; and the *Maxima* in Latitude, as the Sines of the ſame Latitudes directly.

COROL. III.

HENCE may alſo be found the Stars apparent Right Aſcenſion and Declination; for let ECP be the Parallel of the Stars Declination, P the apparent Place of the Star when in that Parallel; make CA perpendicular to CH, ABD to SF, and BE to PC; and let HK, or the Angle HCK be any Diſtance gone over by the Earth in the Ecliptick, while the Star by its apparent Motion moves thro' the correſponding Diſtance PL: Let K*mn*G be parallel to HC, and Lrv to PC: Then, foraſmuch as KL is parallel to HP, the Triangles GKL, CHP muſt be equiangular, and therefore GL : CP :: KL : HP; but KL is to HP, as LI to QP, by the Property of the Curve; whence it will be GL : CP :: LI : QP: Wherefore, the Sides GL, IL, CP, QP about the equal Angles GLI, CPQ being proportional, the Triangles GLI, CPQ muſt be ſimilar, and therefore the Angle GIL a right one, and conſequently the Right Line SF the Locus of the Point G. Therefore, as the Angles n, m, r, v are all given, or continue invariable, let the Angle SCK, or the ecliptick Diſtance of the Sun and Star be what it will, the Ratio of Cm to CG will always be given; but the Ratio of CG to Cr is given; therefore the Ratio of Cm to Cr is likewiſe given: Hence, becauſe rv is parallel to CE, the Ratio of Cm to Ev will be given. But Ev is the Difference of the true and apparent Declinations; and Cm, as the Sine of the Angle HCK: Whence it is manifeſt, that the Aberration of Declination, at any Time, is as the Sine of the Sun's Elongation from either of the two Points wherein he is, when the true and apparent Declinations are the ſame; and therefore Cm will be to Ev, or AC to EB, the greateſt Aberration, as QH to Fh, that is, as the Sine of HCF to the Sine of PCQ: But PCQ, being equal to the Angle of Poſition, is given, whoſe Tangent, it is obvious, is to the Tangent of HCF, as QP to QH, or as CT to CO, or laſtly, (by Conſtruction) as the Sine of the Star's Latitude to Radius: Hence the Angle HCF is given, from which, by Help of the foregoing Theorem or Proportion, the required Aberration of Declination at any Time, and in any Caſe, may be readily obtained.

In like manner other Proportions may be derived for finding the Aberration of Right Aſcenſion; it being eaſy to prove that it will be as the Sine of the Sun's Elongation from where he is, when the true and apparent right Aſcenſions are the ſame; but the Method of Demonſtration being the ſame as above, it will be needleſs to repeat it.

I ſhall therefore now proceed to illuſtrate the foregoing Doctrine by the practical Solutions of the ſeveral Problems depending thereon, as they were drawn up and communicated by Dr. *John Bevis*, with ſuitable Examples of ſeveral Stars, which, among many others, He has carefully obſerved with proper Inſtruments, and thereby, the firſt of any one that I know of, experimentally prov'd, that the Phœnomena are univerſally as conformable to the Hypotheſis in Right Aſcenſions, as the Rev. Mr. *Bradley*, to whom we owe this great Diſcovery, had before found them to be in Declinations.

◆

PRACTICAL RULES
For Finding the
ABERRATIONS
OF THE
FIXT STARS
FROM
The Motion of Light, and of the Earth in its Orbit,
IN
Longitude, Latitude, Declination, and Right Aſcenſion.

SYMBOLS.

A, the Aberration at any given Time.

M, the greateſt Aberration, or *Maximum*.

☉, the Sun's Place in the Ecliptick when the Star's Apparent Longitude, Latitude, Declination, or Right Aſcenſion, being the ſame as the True, tends to Exceſs.

P, the Star's Angle of Poſition. = P*S

Z, the Sun's Elongation from its neareſt Syzygy with the Star, at the Time of ☉.

For

For the Abberration in LONGITUDE.

⊙ is always 3 Signs after the Star's true Place in the Ecliptick.

PROB. I. *To find* M.

RULE.

Coſ. Star's Latit. : *Rad.* :: 20″, 25 : M.

EXAMPLE in γ *Urſæ minoris.*

OPERATION.

Log. *Coſ. Ar. Com.* Star's Latit. 75° 13′ —— 0.5932
+ Log. 20″, 25 —— 1.3064

= Log. M 79″, 36 —— 1.8996

PROB. II. *To find* A.

RULE.

Rad. : *Sin.* Sun's Elongat. from ⊙ :: M : A.

EXAMPLE in the ſame Star.

OPERATION.

Log. *Sin.* Sun's Elongat. from ⊙ 60° 00′ —— 9.9375
+ Log. M 79″,36 —— 1.8996

— Log. *Sin. Rad.* = Log. A 68″,72 —— 11.8371

Otherwiſe, without M.

RULE.

Coſ. Star's Latit. : *Sin.* Sun's Elongat. from ⊙ :: 20″,25 : A.

Same EXAMPLE as before.

OPERATION.

Log. *Coſ. Ar. Com.* Star's Latit. 75° 13′ —— 0.5932
+ Log. *Sin.* Sun's Elongat. from ⊙ 60° 00′ —— 9.9375
+ Log. 20″,25 —— 1.3064

— Log. *Sin. Rad.* = Log. A 68″,72 —— 11.8371

For the Aberration *in* LATITUDE.

⊙ is always at the Sun's Oppoſition to the Star.

PROB. I. *To find* M.

RULE.

Rad. : *Sin.* Star's Latit. : : 20″,25 : M.

EXAMPLE γ in *Urſæ minoris.*

OPERATION.

Log. *Sin.* Star's Latit. 75° 13′ ——— 9.9854
+ Log. 20″,25 ——— 1.3064

— Log. *Sin. Rad.* = Log. M. 19″,58 ——— 11.2918

PROB. II. *To find* A.

RULE.

Rad. : *Sin.* Sun's Elongat. from ⊙ : : M : A.

EXAMPLE in the ſame Star.

OPERATION.

Log. *Sin.* Sun's Elongat. from ⊙ 60° 00′ ——— 9.9375
+ Log. M 19″,58 ——— 1.2918

— Log. *Sin. Rad.* = Log. A 16″,96 ——— 11,2293

Otherwiſe, without M.

RULE.

*Rad*². : *Sin.* Star's Latit. × *Sin.* Sun's Elongat. from ⊙ : : 20″,25 : A.

Same EXAMPLE as before.

OPERATION.

Log. *Sin.* Star's Latit. 75° 13′ ——— 9.9854
+ Log. *Sin.* Sun's Elongat. from ⊙ 60° 00′ ——— 9.9375
+ Log. 20″,25 ——— 1.3064

— 2 Log. *Sin. Rad.* = Log. A 16″,96 ——— 21,2293

Otherwiſe,

R U L E.

Coſec. Star's Latit. : *Sin.* Sun's Elongat. from ⊙ : : 20″,25 : A.

Same EXAMPLE as before.

OPERATION.

Log. *Coſec. Ar. Com.* Star's Latit. 75° 13′	89.9854
+ Log. *Sin.* Sun's Elongat. from ⊙ 66° 00′	9.9375
+ Log. 20″,25	1.3064
= Log. A 16″96	1.2293

For the Aberration *in* DECLINATION.

PROB. I. *To find* ⊙.

R U L E.

Sin. Star's Latit. : *Rad.* : : *Tang.* P. : *Tang.* Z.

Then, if the Star (in reſpect of that Pole of the Equator which is of the ſame Denomination as the Star's Latitude) be in a Sign.

1. Aſcending, and P be acute, Z taken from the oppoſite to its true Place, gives ⊙,
2. Aſcending, and P be obtuſe, Z added to its true Place, gives ⊙,
3. Deſcending, and P be acute, Z added to the oppoſite to its true Place, gives ⊙,
4. Deſcending, and P be obtuſe, Z taken from its true Place, gives ⊙,

provided, that its Declination and Latitude be both North, or both South: But, if one be North, and the other South, then for *its true Place*, read *oppoſite to its true Place*, and *vice verſâ*.

EXAMPLE of *Caſe* I. in the Pole Star.

OPERATION.

Log. *Tang.* P 75° 21′ (acute) + Log. *Sin. Rad.*	20.5827
— Log. *Sin.* Star's Latit. 66° 04′ North.	9.9609
= Log. *Tang.* Z 76° 34′	10.6218

Therefore the Star's Declin. and Latit. being both N. its Place (aſcending) + 6 Signs	8ˢ	24°	55′
— Z	2	16	34
= ⊙.	6	08	21

EXAM-

EXAMPLE of *Caſe* II. in ι *Draconis.*

OPERATION.

Log. *Tang.* P 93° 50′ (obtuſe) + Log. *Sin. Rad.*	21.1739
— Log. *Sin.* Star's Latit. 79° 28′ (North)	9.9926
= Log. *Tang.* Z 86° 14′	11.1813
Therefore the Star's Decl. and Latit. being both North, its true Place (aſcending)	0^s 29° 12′
+ Z	2 26 14
= ⊙	3 25 26

EXAMPLE of *Caſe* III. η *Urſæ majoris.*

OPERATION.

Log. *Tang.* P 38° 36′ (acute) + Log. *Sin. Rad.*	19.9022
— Log. *Sin.* Star's Latit. 54° 25′ (North)	9.9102
= Log. *Tang.* Z 44° 29′	9.9920
Therefore, the Star's Decl. and Latit. being both North, its Place (deſcending) + 6 Signs	11^s 23° 14′
+ Z	1 14 29
= ⊙	1 07 43

EXAMPLE of *Caſe* IV. in γ *Urſæ minoris.*

OPERATION.

Log. *Tang.* P 94° 48′ (obtuſe) + Log. *Sin. Rad.*	21.0759
— Log. *Sin.* Star's Latit. 75° 13′ (North)	9.9854
= Log. *Tang.* Z 85° 22′	11.0905
Therefore, the Star's Decl. and Latit. being both North, its true Place (deſcending)	4^s 17° 50′
— Z	2 25 22
= ⊙	1 22 28

In each of theſe four Examples, the Declination and Latitude are of the ſame Denomination; it may ſuffice to give one where they are of contrary Denominations.

EXAMPLE of *Caſe* III. in *Aldebaran.*

OPERATION.

Log. *Tang.* P 9° 40′ (acute) + Log. *Sin. Rad.*	19.2313
— Log. *Sin.* Star's Latit. 5° 30′ (South)	8.9815
= Log. *Tang.* Z 60° 38′	10.2498

Therefore, Star's Decl. being North, and its Latit. South, its Place (deſcending to S. Pole) —— } 2ˢ 06° 07′
+ Z —— 2 00 38
= ☉ —— 4 06 45′

PROB. II. *To find* M.

RULE.

Sin. Z : *Sin.* P :: 20″,25 : M.

Example in γ *Urſæ minoris.*

OPERATION.

Log. *Sin.* Z *Ar. Com.* 85° 22′ —— 0.0014
+ Log. *Sin.* P 94° 48′ —— 9.9985
+ Log. 20″,25 —— 1.3064
— Log. *Sin. Rad.* = Log. M 20″,24 —— 11.3063

PROB. III. *To find* A.

RULE.

Rad. : *Sin.* Sun's Elongat. from ☉ :: M : A.

Example in η *Urſæ majoris.*

OPERATION.

Log. *Sin.* Sun's Elongat. from ☉ 75° 31′ —— 9.9860
+ Log. M 18″,04 —— 1.2560
— Log. *Sin. Rad.* = Log. A 17″,46 —— 11.2420

Otherwiſe, without M.

RULE.

Rad. × *Sin.* Z : *Sin.* Sun's Elongat. from ☉ × *Sin.* P :: 20″,25 : A.

Same Example as before.

OPERATION.

Log. *Sin.* Sun's Elongat. from ☉ 75° 31′ —— 9.9860
+ Log. *Sin.* P 38° 36′ —— 9.9751
+ Log. *Sin. Ar. Com.* Z 44° 29′ —— 0.1545
+ Log. 20″,25 —— 1.3064
— 2 Log. *Sin. Rad.* = Log. A 17″,46 —— 21.2420

For the Aberration *in* RIGHT ASCENSION.

PROB. I. *To find* ⊙.

RULE.

Sin. Star's Latit. : *Rad.* : : *Cotang.* P. : *Tang.* Z.

Then, if the Star (in reſpect of that Pole of the Equator which is of the ſame Denomination as the Star's Latitude) be in a Sign

1. Aſcending, and P be acute, Z added to its true Place, gives ⊙.
2. Aſcending, and P be obtuſe, Z taken from its true Place, gives ⊙.
3. Deſcending, and P be acute, Z taken from the oppoſite to its true Place, gives ⊙.
4. Deſcending, and P be obtuſe, Z added to the oppoſite to its true Place, gives ⊙.

EXAMPLE of *Caſe* I. in *Sirius.*

OPERATION.

Log. *Sin. Ar. Com.* Star's Latit. 39° 32′ (South.)			0.1962
+ Log. *Cotang.* P 4° 18′ (acute)			11.1238
= Log. *Tang.* Z 87° 16′			11.3200
Therefore the Star's true Place (aſcending)	3^{s}	10°	29′
+ Z	2	27	16
= ⊙	6	07	45

EXAMPLE of *Caſe* II. in • *Draconis.*

OPERATION.

Log. *Sin. Ar. Com.* Star's Latit. 79° 28′ (North)			0.0074
+ Log. *Cotang.* P 93° 50′ (obtuſe)			8.8261
= Log. *Tang.* Z 3° 54′			8.8335
Therefore the Star's true Place (aſcending)	0^{s}	29°	12′
− Z	0	03	54
= ⊙	0	25	18

EXAMPLE of *Caſe* III. γ *Draconis.*

OPERATION.

Log. *Sin. Ar. Com.* Star's Latit. 74° 28′ (North)	0.0162
+ Log. *Cotang.* P 3° 36′ (acute)	11.2012
= Log. *Tang.* Z 86° 32′	11.2174
Therefore, the Star's true Place (deſcending) + 6 Signs	8^{s} 24° 23′
— Z	2 26 32
= ⊙	11 27 51

EXAMPLE of *Caſe* IV. in γ *Urſæ minoris.*

OPERATION.

Log. *Sin. Ar. Com.* Star's Latit. 75° 13′ (North)	0.0146
+ Log. *Cotang.* P 94° 48′ (obtuſe)	8.9241
= Log. *Tang.* Z 4° 58′	8.9387
Therefore, the Star's true Place (deſcending) + 6 Signs	10^{s} 17° 50′
+ Z	0 04 58
= ⊙	10 22 48

PROB. II. *To find* M.

RULE.

Coſ. Star's Decl. × *Sin.* Z : *Coſ.* P × *Rad.* :: 20″,25 : M.

EXAMPLE in the Pole Star.

OPERATION.

Log. *Coſ. Ar. Com.* Decl. 87° 55′	1.4395
+ Log. *Sin. Ar. Com.* Z 15° 58′	0.5605
+ Log. *Coſ.* P.	9.4030
+ Log. 20″,25	1.3064
— Log. *Sin. Rad.* = Log. M. 512″,16 = 8′. 32″,16	12.7094

PROB. III. *To find* A.

RULE.

Rad. : *Sin.* Sun's Elongat. from ☉ :: M : A.

EXAMPLE in *Lucida Aquilæ*.

OPERATION.

Log. *Sin.* Sun's Elongat. from ☉ 65° 24′	9.9587
+ Log. M 20″,18	1.3049
— Log. *Sin. Rad.* = Log. A 18″,34	11.2636

Otherwiſe, without M.

RULE.

Coſ. Star's Decl. × *Sin.* Z : *Sin.* Sun's Elongat. from ☉ × *Coſ.* P :: 20″,25 : A.

Same EXAMPLE.

OPERATION.

Log. *Sin.* Sun's Elongat. from ☉ 65° 24′	9.9587
+ Log. *Coſ. Ar. Com.* Star's Decl. 8° 12′	0.0045
+ Log. *Sin. Ar. Com.* Z 84° 36′	0.0019
+ Log. *Coſ.* P 10° 55′	9.9921
+ Log. 20″,25	1.3064
— 2 Log. *Rad.* = Log. A 18″,34	21.2636

GENE-

GENERAL NOTES.

1. That the Rules give the Values of A and M, always in Seconds of a Degree.

2. That if the Sun's Place be in that Semicircle of the Ecliptic which precedes ☉, A muſt be taken from the Star's True Longitude, Latitude, Declination, or Right Aſcenſion, to ſhew the Apparent; but if it be in that Semicircle which follows ☉, A muſt be added.

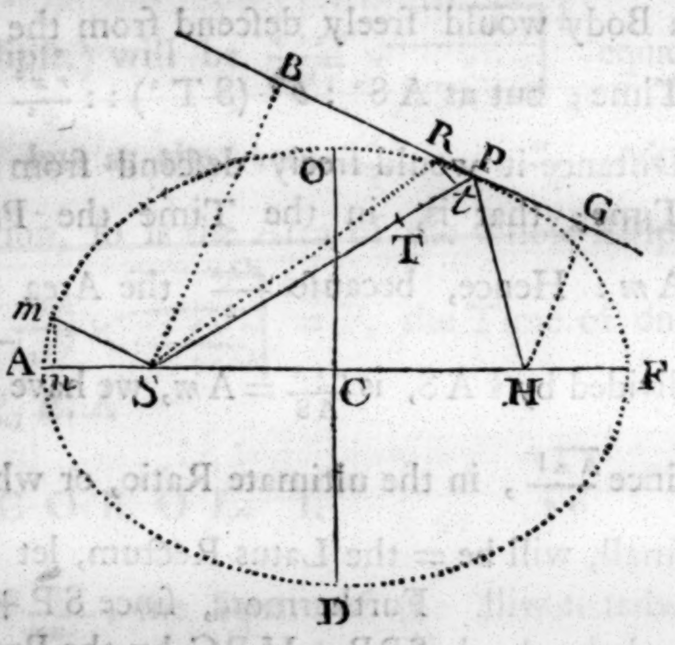
3. That ♑, ♒, ♓, ♈, ♉, ♊ are Signs Aſcending in reſpect of the North Pole, and Deſcending in reſpect of the South Pole of the Equator: And ♋, ♌, ♍, ♎, ♏, ♐, are Aſcending in reſpect of the South Pole, and Deſcending in reſpect of the North Pole of the Equator.

4. That a Star may be ſo poſited, that the ſmall Ellipſe which it apparently deſcribes, may, by including, or approaching very near to the Pole of the World, make it fall under very different Conſiderations and Rules from any of the foregoing; but as the beſt Inſtruments have not diſcovered any ſuch Stars, thoſe Conſiderations and Rules have been here omitted.

Of the MOTION *and* ORBITS *of Bodies affected with Projectile and Centripetal Forces.*

PROPOSITION I.

A Body being let go from P, *at a given Diſtance* PS, *from* S, *the Center of Force, in a given Direction* P B, *with a given Velocity; To find the Conic Section it will deſcribe, and the Periodic Time, in caſe it returns, the Law of Centripetal Force being as the Square of the Diſtance inverſely, and the abſolute Force given.*



LET ASF be the greater Axis of the Section, OCD the leſſer, and H the upper Focus. Suppoſe SR indefinitely near SP, and the Area A S*m* equal to the Area PSR; draw S B and H G perpendicular to the given Tangent BPG, and R*t* and *m n* to S P and A F reſpectively: Let *r* be the Diſtance that a Body would freely deſcend in any given Time, *m*, by an uniform Force, equal to that affecting the Projectile at any given Diſtance S T (*b*) from the Centre, and let *v* be the Space that the Body would uniformly deſcribe with the given Velocity at P, in the ſame Time: Call A F, *a*; O D, *e*; the Latus

Rectum, or $\frac{ee}{a}$, R; S P, d; P R, x; the Periodic Time, P, and the Sine of S P B, to the Radius 1, s: Then it will be as $1 : s :: x : sx = Rt$; whence $\frac{sxd}{2}\left(= Rt \times \frac{SP}{2}\right)$ will expreſs either of the infinitely ſmall equal Areas PSR, mSA: And it will be as $v : m :: x : \frac{mx}{v}$ the Time of the Projectile's moving thro' P R, or that of its deſcribing either of the ſaid Areas by Radii drawn to the Centre of Force: Wherefore, the Diſtances which Bodies freely deſcend by uniform Forces, being as the Squares of the Times, we have, as $m^2 :: r :: \frac{m^2x^2}{v^2}$ the Square of that Time, to $\frac{rx^2}{v^2}$ the Diſtance a Body would freely deſcend from the Point T in the ſame Time; but as $AS^2 : b^2\ (ST^2) :: \frac{rx^2}{v^2} : \frac{rb^2x^2}{v^2 \times AS^2} = An$, the Diſtance it would freely deſcend from the Point A in that Time; that is, in the Time the Projectile is deſcribing Am: Hence, becauſe $\frac{dsx}{2}$ the Area of the Triangle ASm; divided by $\frac{1}{2}$ A S, is $\frac{dsx}{AS} = Am$, we have $\frac{\overline{Am}|^2}{An} = \frac{d^2v^2s^2}{rbb} = R$, ſince $\frac{\overline{Am}|^2}{An}$, in the ultimate Ratio, or when An is indefinitely ſmall, will be = the Latus Rectum, let the Curve A P F be what it will. Furthermore, ſince $SP + PH$ is $= a$, or A F, and the Angle $SPB = HPG$ by the Property of the Curve, it will be as $1 : s :: d : sd = SB$, and as $1 : s :: a - d : s \times \overline{a - d} = HG$; but by another Property, $BS \times HG$ is $= OC^q$ or $ss \times \overline{ad - dd} = \frac{ee}{4}$; whence by ſubſtituting this Value of $\frac{ee}{4}$, in the other Equation $\frac{d^2v^2s^2}{rbb} = \frac{ee}{a}$ (= R) we get $a = \frac{d}{1 - \frac{dv^2}{4rbb}}$, and therefore $e = \frac{a^{\frac{1}{2}} v s d}{b\sqrt{r}} = \left.\overline{\frac{a}{1-\frac{dvv}{4rbb}}}\right|^{\frac{1}{2}} \times \frac{vsd}{b\sqrt{r}}$, and $PH = \frac{d}{1-\frac{dvv}{4rbb}} - d$, from which, as the Angle H P G is given, the Focus H is given likewiſe; whence the Orbit may be readily conſtructed, it being, when $\frac{d}{1-\frac{dvv}{4rbb}}$ is poſitive, an Ellipſis, when infinite, a Parabola, and when negative, an Hyperbola; wherefore, unleſs $1-\frac{dv^2}{4rbb}$ be affirmative, or $\frac{4rbb}{d}$ greater than vv, the Projectile can never return. Now, therefore, putting p for the Area of a Circle whoſe Diameter is Unity, and ſuppoſing $\frac{4rbb}{d}$ greater than vv, the Area of the whole Curve, (being an Ellipſe,) will be $\frac{vsdp}{b\sqrt{r}} \times \left.\overline{\frac{d}{1-\frac{dvv}{4rbb}}}\right|^{\frac{3}{2}}$ equal to $\frac{vsdpa^{\frac{3}{2}}}{b\sqrt{r}}$ $(= pae)$; but as the Area $\frac{sdx}{2}$ is to $\frac{mx}{v}$ the Time of its Deſcription, ſo is the Area of the whole Ellipſis, to $\frac{2pm}{b\sqrt{r}} \times a^{\frac{3}{2}}$ or $\frac{2pm}{b\sqrt{r}} \times \left.\overline{\frac{d}{1-\frac{dvv}{4rbb}}}\right|^{\frac{3}{2}} = P$, the Time of one intire Revolution. *Q. E. I.*

COROL. I.

BECAUSE $\frac{m^2x^2}{vv}$, the Square of the Time of deſcribing the Area RSP, is to $\frac{s^2d^2x^2}{4}$, the Square of that Area, as (1) the Square of a conſtant Particle of Time to $\frac{v^2s^2d^2}{4m^2}$ the Square of the Area deſcribed in this laſt Time, it is evident that the Square laſt named will be to the Latus Rectum $\frac{v^2 s^2 d^2}{r b^2}$ as $r b^2 : m^2$; which Proportion being conſtant in all Caſes relating to the ſame Center, it follows, that the principal Latera Recta of the Orbits of different Bodies, about a common Centre of Force, are directly as the Squares of the Areas deſcribed by the reſpective Bodies, in the ſame Time.

COROL. II.

MOREOVER, ſince B S is $= s d$, and $R = \frac{d^2 v^2 s^2}{r b b}$, we have $\frac{R^{\frac{1}{2}}}{BS} = \frac{v}{b \sqrt{r}}$ and $\because$ $\frac{R^{\frac{1}{2}}}{SB}$ is to v, in the conſtant Ratio of 1, to $\frac{1}{b\sqrt{r}}$: Hence it appears, that the Velocities are, univerſally, in the ſubduplicate Ratio of the Parameters directly, and the Perpendiculars falling from the Center of Force on Tangents to the Places of the Bodies, inverſely, and therefore, in the ſame Orbit, the Velocity will be, barely, in the inverſe Ratio of the Perpendiculars ſo falling.

COROL. III.

SINCE P is $= \frac{2pm}{br^{\frac{1}{2}}} \times a^{\frac{3}{2}}$, or, in a conſtant Proportion, to $a^{\frac{3}{2}}$, let v, s, and d, be what they will; it follows, that the Periodic Times, about the ſame Center of Force, whether in Circles or Ellipſes, will be in the ſeſquiplicate Ratio of the principal Axes.

COROL.

COROL. IV.

BECAUSE neither the Values of a nor P are affected by s, it follows, that the principal Axis, and the Periodic Time will be the ſame, if the Velocity at P be the ſame, let the Direction of the Projectile at that Point be what it will.

COROL. V.

WHEN $\frac{d}{1-\frac{dvv}{4rbb}}$ $(=a)$ is $=2d$, or, which is the ſame, when $v=\sqrt{\frac{2r}{d}}$; then d being the mean Diſtance or Semi-Tranſverſe, the Point P will fall in one Extreme of the Conjugate Axis, and $b\sqrt{\frac{2r}{d}}$, the Velocity there, will be juſt ſufficient to retain a Body in a Circular Orbit at that Diſtance (d) from the Center of Force; and this Velocity, in reſpect of different Orbits, will, it is obvious, be inverſely as the Square Roots of the mean Diſtances: Wherefore the Velocities Bodies in Circular Orbits about a common Centre, are reciprocally in the ſubduplicate Ratio of the Radii.

COROL. VI.

IF v be $=b\sqrt{\frac{4r}{d}}$, or the Square of the Velocity be juſt twice as great as that whereby the Projectile might deſcribe a circular Orbit at its own Diſtance from the Center of Force (*Cor.* V.) then a, the Tranſverſe, becoming infinite, the Ellipſe degenerates into a Parabola, whoſe principal *Latus Rectum* is $4dss$; whence it appears, that the Velocity of a Body moving in a Parabola is inverſely as the Square Root of its Diſtance from the Centre of Force, and that it will be, every where, to the Velocity

locity that might carry the Projectile in a circular Orbit, at its own Diſtance from the Centre, as the Square Root of two, to one.

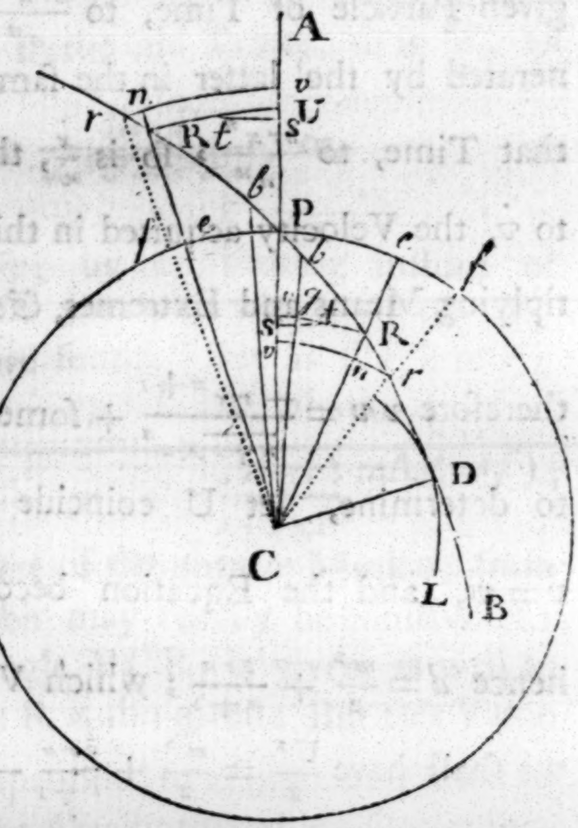
COROL. VII.

BUT if v be greater than $b\sqrt{\frac{4r}{d}}$, the Trajectory will be an Hyperbola, whoſe principal Axis is $\frac{d}{\frac{dvv}{4rbb}-1}$ $(=-a)$ as has been before intimated, and therefore e $\left(=\frac{a^{\frac{1}{2}}vsd}{b\sqrt{r}}\right)$ will be $\frac{2vsd^{\frac{3}{2}}}{\sqrt{dv^2-4rbb}}$. Hence, from the Nature of the Hyperbola, if R be aſſumed for Radius, $\frac{vsd^{\frac{1}{2}}R\times\sqrt{dvv-4rbb}}{2rbb}$ $\left(=\frac{eR}{-a}\right)$ will be the Tangent of the Angle which the Aſymptote makes with the Axis, or the Supplement to 180° of the utmoſt Elongation the Projectile can poſſibly have from the loweſt Point of its Orbit.

PROP.

PROPOSITION II.

A Body is let go from P, *at a given Diſtance* P C *from* C *the Centre of Force, in a given Direction* P *b*, *with a given Celerity; To find its Trajectory? the centripetal Force being as any Power* (*n*) *of the Diſtance, and the abſolute Force at* P *given.*



LET R be a Point in the required Trajectory, and r another indefinitely near it; and with the Centre C, let the Circular Arches P ef, R U, vnr, be deſcribed, and having drawn C R e, C rf, &c. let C P $= a$, C R ($=$ C U) $= x$, P $e =$ A, R n ($=$ U v) $= \dot{x}$, $rn = \dot{y}$, Rr $= \dot{z}$, and $s =$ the Sine of the Angle C P b to the Radius 1; and let P b (m) be the Space that might be deſcribed in (1) a given Particle of Time with the given Celerity; and r the Diſtance a Body would freely deſcend in that Time, by an uniform Force equal to that affecting the Projectile at P: Then the Space which would be uniformly deſcribed in that ſame Time, with the Celerity acquired by deſcending thro' the ſaid Diſtance r, it is well known, will be equal to 2 r. But, from hence to find the Celerity at R, with the ſame Velocity that the given Projectile is let go from P, towards b, let another proceed from the ſame Point, in a Right-Line paſſing directly thro' the Center of Force; and let the Celerity at U, or the Space that would be uniformly deſcribed therewith in 1, the aboveſaid Particle of Time, be denoted by v: Then, as a^n, the Centripetal Force at P, is to x^n, that at U, ſo is $2r$, the Velocity that might be generated by the former in the given Particle of Time, to $\frac{2rx^n}{a^n}$, that which would be generated by the latter in the ſame Time: Wherefore, as 1, that Time, to $\frac{2rx^n}{a^n}$, ſo is $\frac{\dot{x}}{v}$, the Time of deſcribing U v, to $\dot{v}$, the Velocity acquired in this Time: Whence, by multiplying Means and Extremes, *&c.* we get $v\dot{v} = \frac{2rx^n\dot{x}}{a^n}$, and therefore $vv = \frac{-rx^{n+1}}{\overline{n+1} \times a^n}$ + ſome conſtant Quantity d; which to determine, let U coincide with P, x be $= a$, and $v = m$, and the Equation becomes $\frac{m^2}{2} = \frac{-2ra^{n+1}}{\overline{n+1} \times a^n} + d$; hence $d = \frac{m^2}{2} + \frac{2ra}{n+1}$; which Value being ſubſtituted above, we ſhall have $\frac{U^2}{2} = \frac{m^2}{2} + \frac{2ra}{n+1} - \frac{2rx^{n+1}}{\overline{n+1} \times a^n}$; and therefore U $= \left|\overline{m^2 + \frac{4ra}{n+1} - \frac{4rx^{n+1}}{\overline{n+1} \times a^n}}\right|^{\frac{1}{2}}$: But this is likewiſe the Celerity of the firſt Projectile at R: For ſince both Bodies have the ſame Velocity at P, their Velocities, at all equal Diſtances from the Centre muſt be equal; and therefore $\sqrt{m^2 + \frac{4ra}{n+1} - \frac{4rx^{n+1}}{\overline{n+1} \times a^n}}$, or its Equal $\frac{\sqrt{\dot{x}\dot{x}+\dot{y}\dot{y}}}{v}$ will conſequently be the Time of the ſaid given Projectiles moving thro' Rr, or of deſcribing the the Area $\frac{x\dot{y}}{2}=$ RCr, by Radii drawn to the Centre of Force: Wherefore, ſince $\frac{sma}{2}$ is the Area of the little Triangle PCb that might be uniformly deſcribed in, 1, the given Particle of Time, with the Velocity at P; and, becauſe the Areas are as the Times, it will be, as $\frac{sma}{2}$ to 1 (the ſaid Time), ſo is $\frac{x\dot{y}}{2}$, to $\frac{\sqrt{\dot{x}^2+\dot{y}^2}}{v}$: Hence we get $\dot{y} = \frac{sma\dot{x}}{\sqrt{a^2v^2-m^2s^2a^2}}$, or, $\frac{sma\dot{x}}{\sqrt{m^2x^2+\frac{4rax^2}{n+1}-m^2s^2a^2-\frac{4rx^{n+3}}{\overline{n+1}\times a^n}}}$, by ſubſtituting inſtead of v^2, its known Value, as above found. But as Cr, is to rn, :: Cf $(=a)$: $\frac{smaa\dot{x}}{x\sqrt{m^2x^2+\frac{4rax^2}{n+1}-m^2s^2a^2-\frac{4rx^{n+3}}{\overline{n+1}\times a^n}}} = \dot{A}$ $(=ef)$; whoſe Fluent Pe is the Meaſure of the angular Motion; from which, when found, the Orbit may readily be conſtructed; becauſe, when Pe, or the Angle PCR, is given, as well as CR, the Poſition of the Point R is alſo given: But this Value of $\dot{A}$ is indeed too much compounded to admit of a Fluent in general Terms, or even by the Quadrature of the Conic Sections, except in certain particular Caſes, as where n is equal to 1, —2, —3, or —5, or the Law of centripetal Force, as the firſt Power of the Diſtance directly, or the 2d, 3d, or 5th Powers thereof inverſely; therefore, in other Caſes, can only be had by infinite Series, *&c.* or Curves of a ſuperior Order. *Q. E. I.*

COROL. I.

IF, inſtead of the abſolute Celerity of the Projectile at P, the Ratio thereof to that which it ſhould have to deſcribe the Circle P*e*, be given, as p to 1, and not only the ſame Thing, but the Ratio between the Celerity at any other Diſtance CR, and that which a Body muſt have to deſcribe a circular Orbit at that Diſtance, be required: It will be, as a^n, the centripetal Force at P, to x^n, that at R (or U), ſo is r, the Diſtance a Body would freely deſcend by the former of theſe Forces in 1, the given Particle of Time, to $\frac{rx^n}{a^n}$, that which it would deſcend by the latter in the ſame Time: Therefore, if Us be taken equal to $\frac{rx^n}{a^n}$, and st be made perpendicular to AC, it is manifeſt, that Ut, being indefinitely ſmall, will be the Diſtance which a Body muſt move over in the aforeſaid Particle of Time, to deſcribe the Circle UR: But Ut, by the Property of the Circle, is in that Circumſtance $= \sqrt{\frac{2rx^{n+1}}{a^n}}$; wherefore we have, as $\sqrt{\frac{2rx^{n+1}}{a^n}}$, to v, or its Equal, $\sqrt{m^2+\frac{4ra}{n+1}-\frac{4rx^{n+1}}{n+1}}$ (above found), ſo is the Velocity a Body muſt have to deſcribe that Circle, to that with which the given Projectile arrives at R: Therefore, when x is $=a$, and R coincides with P, the Proportion of $\sqrt{\frac{2rx^{n+1}}{a^n}}$, to $\sqrt{m^2+\frac{4ra}{n+1}-\frac{4rx^{n+1}}{\overline{n+1}\times a^n}}$, which there is, as $\sqrt{2rn}$, to m, is given as 1, to p, by Suppoſition; whence, multiplying Extremes

tremes and Means, we get $m=p\sqrt{2ra}$; which being ſubſtituted inſtead of m, in the Value of v, it will become $\sqrt{2p^2ra+\frac{4ra}{n+1}-\frac{4rx^{n+1}}{\overline{n+1}\times a^n}}$ or $\overline{p^2+\frac{2}{n+1}-\frac{2}{n+1}\times\frac{x^{n+1}}{a^{n+1}}}\Big|^{\frac{1}{2}}\times\overline{2ra}\,\Big|^{\frac{1}{2}}$; therefore this divided by $\sqrt{\frac{2rx^{n+1}}{a^n}}$, is $\sqrt{\overline{p^2+\frac{2}{n+1}}\times\frac{a^{n+1}}{x^{n+1}}-\frac{2}{n+1}}$, for the Ratio that was to be found. And, in like manner, by ſubſtituting for m in the Value of $\dot{A}$, we get $\frac{spaa\dot{x}}{x\sqrt{\overline{p^2+\frac{2}{n+1}}\times x^2-p^2s^2a^2-\frac{2x^{n+3}}{\overline{n+1}\times a^{n+1}}}}$ for the other Quantity required.

C O R O L. II.

HENCE, if the Angle CPb be ſuppoſed to be diminiſhed *in infinitum*, and $\overline{p^2+\frac{2}{n+1}-\frac{2}{n+1}\times\frac{x^{n+1}}{a^{n+1}}}\Big|^{\frac{1}{2}}\times\overline{2ra}\,\Big|^{\frac{1}{2}}$, the ſaid Value of v, be taken $=0$, we ſhall have $x=\overline{\tfrac{1}{2}pp\times\overline{n+1}+1}\Big|^{\frac{1}{n+1}}\times a$, ($=$CA) the Height to which the Body would aſcend, if projected directly upwards; therefore, $\overline{\tfrac{1}{2}pp\times\overline{n+1}+1}\Big|^{\frac{1}{n+1}}\times a-a=$AP, is the Diſtance it muſt freely deſcend to acquire the given Velocity; which Diſtance, therefore, with an uniform Centripetal Force, where $n=0$, will be $=\frac{ppa}{2}$; and with a Force inverſely, as the Square of the Diſtance, $=\frac{ppa}{2-pp}$. But when p is $=1$, or the Velocity of the Projectile at P is juſt ſufficient to retain a Body in the circular Orbit P e, A P then becomes $\overline{\frac{3+n}{2}}\Big|^{\frac{1}{n+1}} \times a - a$; which in the ſaid two Caſes, will be $\frac{1}{2}a$, and a reſpectively; but infinite when n is $= -3$.

COROL. III.

WHEN $n+1$ is a poſitive Number, the Velocity $\sqrt{2ra} \times \overline{p^2 + \frac{2}{n+1} - \frac{2}{n+1} \times \frac{x^{n+1}}{a^{n+1}}}\Big|^{\frac{1}{2}}$, at the Centre C, where x becomes $= 0$, will, it appears, be barely equal to $\sqrt{2ra \times \overline{p^2 + \frac{2}{n+1}}}$; but, when $n+1$ is negative, or the Law of Centripetal Force more than the firſt Power of the Diſtance inverſely, it will be infinite; becauſe then, the Index being negative, x^{n+1} (or its Equal 0^{n+1}) will come into the Denominator.

COROL. IV.

MOREOVER, when $n+1$ is negative, and x infinite, the ſaid Velocity will alſo become $\sqrt{2ra \times \overline{p^2 + \frac{2}{n+1}}}$; becauſe then, for the Reaſon above ſpecified, x^{n+1} will be $= 0$: And therefore, when the Centripetal Force is more than the firſt Power of the Diſtance inverſely, a Projectile moving from P with the given Velocity $p\sqrt{2ar}$ $(= m)$ along the Right-Line P A, will aſcend even to an infinite Height, and have a Velocity there ſignified by $\sqrt{ra \times \overline{p^2 + \frac{2}{n+1}}}$, or in Proportion to the given Velocity, as $\sqrt{p^2 + \frac{2}{n+1}}$, to p, provided $p^2 + \frac{2}{n+1}$

be poſitive; for otherwiſe the Thing is impoſſible, the Square Root of that Quantity being manifeſtly ſo.

COROL. V.

HENCE, if the leaſt Velocity that can carry the Body to an infinite Height, or that which it would acquire by freely deſcending from the ſame Height, be required: By making $p^2+\frac{2}{n+1}=0$, we ſhall have $p=\sqrt{\frac{-2}{n+1}}$; which, ſubſtituted in $p\sqrt{2ar}$ gives $\sqrt{\frac{-2}{n+1}}\times\sqrt{2ar}=2\sqrt{\frac{-ra}{n+1}}$, for the Value ſought; and this, it is manifeſt, is to $\sqrt{2ar}$, the Velocity a Body muſt have to deſcribe the Circle Pe, as $\sqrt{\frac{-2}{n+1}}$, to Unity: Therefore, when n is leſs than -3, or the Law of Centripetal Force more than the Cube of the Diſtance inverſely, a leſs Velocity will carry a Projectile to an infinite Height in a Right-Line, than can retain it in a circular Orbit, was it turned into a proper Direction.

COROL. VI.

WHEREFORE, if it were required, how far a Body muſt deſcend by an uniform Force equal to that affecting the Projectile at the Point P, to acquire the ſame Celerity that another Body, by freely falling from an infinite Height (as above) has at its Arrival to that Point; then, by ſubſtituting the Value $\sqrt{\frac{-2}{n+1}}$, as found in the laſt Article, inſtead of its Equal, in $\frac{ppa}{2}$ (ſee *Cor.* II.) there comes out $\frac{-a}{n+1}$ for the Value ſought: And hence it appears, that the Velocity with which a Body, falling freely from an infinite Height, would impinge on the Earth, is no greater than that which another Body may acquire by an uniform Gravity, equal to that at its Surface, in falling freely thro' a Space equal to its Semi-diameter.

SCHOLIUM.

FROM the Ratio found in *Corollary* I. between the Velocity with which the Body arrives at any Diſtance (x) from the Centre of Force, and that which it ought to have to deſcribe a Circle at the ſame Diſtance, it will not be difficult to determine in what Caſes the Body will be compelled to fall to the Centre, and in what other Caſes it will fly *ad infinitum* therefrom. For, firſt, if the Body in moving from P, begins to deſcend, or the Angle CPb be acute, I ſay, it will continue to do ſo 'till it actually falls into the Centre of Force, if the Quantity $\left(\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}\right)$ in its Acceſs thereto, be not ſomewhere greater than Unity; or, which is the ſame in effect, unleſs the Body has ſomewhere a Velocity more than ſufficient to retain it in a circular Orbit at its own Diſtance from the Center of Force: For, if it ever begins to aſcend, it muſt be at a Point, as D, where a Right-Line, drawn from the Centre, cuts the Orbit perpendicularly, and there, it is manifeſt, the Celerity muſt be as above ſpecified, otherwiſe the Body will ſtill continue to deſcend, or elſe move in the Circle DL about the Center C, which is equally abſurd. On the contrary, if the ſaid Quantity, in approaching the Centre, increaſes ſo as to become greater than Unity, or be every where ſo; then, the Velocity at all inferior Diſtances, being greater than the Velocity that is ſufficient to retain a Body in a circular Orbit at any ſuch Diſtance, the Projectile cannot, it is evident, be forced to the Centre.

But, on the other hand, the Angle C P b, being ſuppoſed obtuſe, it will evidently appear from a like Reaſoning, that, if the ſaid Quantity be always greater than Unity, or the Body in its Receſs from the Center, has, in every Place thro' which it paſſeth, a Velocity greater than is ſufficient to retain it in a circular Orbit at the Diſtance of that Place from the Center of Force, it muſt, of conſequence, continue to aſcend *ad infinitum*.

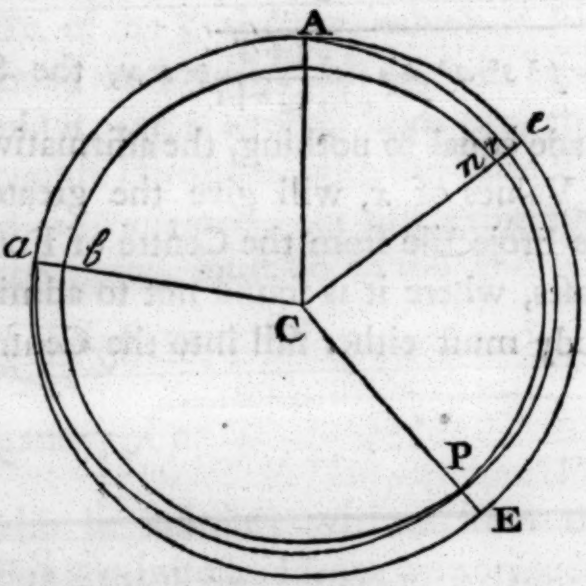
Now, therefore, to find in what Laws of Centripetal Force theſe different Caſes obtain, let the Angle C P b be firſt ſuppoſed acute, or the Body moving towards the Centre, and x in the aboveſaid Quantity $\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}$, to be infinitely ſmall; then it is evident, that that Quantity will become either $\sqrt{\frac{-2}{n+1}}$, or *infinite*, according as $n+1$ is a negative Number, or otherwiſe; wherefore, in the latter of theſe two Caſes, the Body can never be forced into the Centre; neither can it in the former, when n has any Value betwixt -1 and -3, as is manifeſt from above, becauſe $\sqrt{\frac{-2}{n+1}}$ is greater than Unity (Rectilinear Motion being here excepted:) Nor will either of theſe Concluſions hold leſs true, when the Angle C P b is obtuſe; for it is obvious, that if the Projectile cannot be forced to the Centre, when directed towards it with the leaſt Obliquity, it never can, when the Obliquity is increaſed: But on the contrary, if $n+1$ be either equal to or leſs than -2, and p be leſs than 1; then the ſaid Value $\sqrt{\frac{-2}{n+1}}$ not being greater than Unity, the Projectile muſt inevitably be drawn into the Centre; for, the afore-mentioned general Expreſſion not exceeding Unity, neither at the given Diſtance a, nor at the leaſt aſſignable Diſtance, cannot at an intermediate Diſtance; becauſe, in the Deſcent of the Body, the Expreſſion muſt either increaſe or decreaſe continually, there being only one Dimenſion of the variable Quantity (x) concerned. But, when p is greater than Unity, other Things continuing the ſame, I ſay, the Body, if it eſcapes the Centre, and once begins to aſcend, it will continue to fly from the ſame *ad infinitum*. For, ſince the Part D L, *&c.* of the Trajectory, which it will begin to deſcribe on its leaving the loweſt Point D, is in every reſpect equal and ſimilar to D R, *&c.* if another Body projected upwards from P, in the oppoſite Direction, with the ſame Velocity, continues to aſcend *ad infinitum*, our firſt Projectile, after it has paſſed the loweſt Point, muſt do ſo too, and *vice verſa*; therefore $p^2 + \frac{2}{n+1}$ being there affirmative, and the Angle CPb obtuſe, the Quantity $\sqrt{p^2 + \frac{2}{n+1} \times \frac{a^{n+1}}{x^{n+1}} - \frac{2}{n+1}}$, when x is infinite, will alſo be infinite; whence from the above Reaſoning, the Poſition is manifeſt. Hence we conclude, *firſt*, that when n is greater than -3, or the Law of Centripetal Force, as any Power of the Diſtance directly, or leſs than the Cube thereof inverſely, the Body cannot poſſibly fall into the Centre, except in a Right-Line. And, *ſecondly*, that, when the Force is, as the Cube, or more than the Cube of the Diſtance inverſely, it muſt either be forced to the Centre, or fly an infinite Diſtance therefrom, unleſs it moves in a Circle.

Furthermore, becauſe the aboveſaid Quantity, when x is infinite, in all Caſes where $n+1$ is negative, and pp greater than $\frac{-2}{n+1}$, appears to be greater than Unity, it follows, that in all thoſe Caſes, the Body may aſcend, even to an infinite Height, and actually will do ſo, when n has any Value betwixt -1 and -3; becauſe then, tho' the Body ſhould at firſt approach towards the Centre, its Aſcent cannot be anticipated by being drawn into it, as it may, when the Value of n is ſmaller, as has been above ſhewn.

Note, The ſame Things may be otherwiſe determined by Help of the laſt general Value of $\dot{A}$; for if $\overline{\overline{pp+\frac{2}{n+1}}\times xx - p^2 s^2 a^2 - \frac{2x^{n+3}}{n+1\times a^{n+1}}\times xx}$, the Square of its Diviſor be made equal to nothing, the affirmative Roots of that Equation, or Values of x, will give the greateſt and leaſt Diſtances of the Projectile from the Centre of Force, and therefore in thoſe Caſes, where it is found not to admit of two ſuch Roots, the Body muſt either fall into the Centre, or fly it *ad infinitum*.

PROPOSITION III.

To find the Motion, or Angular Diſtance of the Apſides, in Orbits nearly circular; the centripetal Force being as any Power of the Diſtance.



LET A *r* P *a* be the propoſed Orbit, A and P two Places of the higher and lower *Apſides*, A *e* E *a* A, and *n*P*bn*, Circles deſcribed with the Radii A C, C P about C, the Centre of Force; let *r* be a Point in the Trajectory taken at Pleaſure; and let the Velocity of the Body at the higher *Apſe* be to that which it ought to have to retain itſelf in the Circle A *e* E, as $\sqrt{1-e}$, to 1; calling AC, 1; *re*, *y*; C *r*, $1-y$; and A *e*, A: Then, by ſubſtituting, 1 for *a*, 1 for *s*, $\sqrt{1-e}$ for *p*, $1-y$ for *x*, and $\dot{y}$ for $\dot{x}$, in $\frac{spaa\dot{x}}{x\sqrt{\overline{p^2+\frac{2}{n+1}}\times xx-p^2s^2a^2-\frac{2x^{n+3}}{\overline{n+1}\times a^{n+1}}}}$ the general Value of $\dot{A}$, as found by the laſt *Prob.* the ſame it is manifeſt, will become $\frac{\sqrt{1-e}\times\dot{y}}{\overline{1-y}\sqrt{\overline{1-e+\frac{2}{n+1}}\times\overline{1-y}|^2-1+e-\frac{2\times\overline{1-y}|^{n+3}}{n+1}}}$ for the Value of $\dot{A}$ in this particular Caſe; which by reducing $\overline{1-y}|^2$ and $\overline{1-y}|^{n+3}$ into ſimple Terms, is

$$\frac{\dot{y}\sqrt{1-e}}{\overline{1-y}\sqrt{1-e+\frac{2}{n+1}\times\overline{1-2y+yy}-1+e-\frac{2}{n+1}\times 1-\overline{n+3}\times y+n+3\times\frac{n+2}{2}y^2},}$$

$$\&c. = \frac{\dot{y}\sqrt{1-e}}{\overline{1-y}\sqrt{2ey-ey^2-\overline{n+3}\times yy-\frac{2}{n+1}\times\frac{n+3}{1}\times\frac{n+2}{2}\times\frac{n+1}{3}y^3,\ \&c.}}$$

but, becauſe e and y, by the Nature of the Queſtion, are very ſmall, all the Terms wherein more than two Dimenſions of theſe Quantities are concerned, may be rejected as inconſiderable in reſpect of the reſt; by doing which, our Equation becomes $\dot{A} = \frac{\dot{y}\sqrt{1-e}}{\overline{1-y}\sqrt{2ey-\overline{n+3}\times yy}}$; which (for the above Reaſon) is $= \frac{\dot{y}}{\sqrt{2ey-\overline{n+3}\times yy}}$, or $\frac{1}{\sqrt{n+3}}\times\frac{\dot{y}}{\sqrt{\frac{2ey}{n+3}-yy}}$ very nearly: But the Fluent of $\frac{\dot{y}}{\sqrt{\frac{2ey}{n+3}-yy}}$ when $\sqrt{\frac{2ey}{n+3}-yy}$ becomes $=0$, or $A=AeE$, is equal to a Semi-circle whoſe Radius is Unity, or to 180 Degrees; therefore $\frac{1}{\sqrt{n+3}}\times 180° = \frac{180}{\sqrt{n+3}}$ Degrees, is the Meaſure of the Angle ACP. *Q. E. I.*

COROL. I.

WHEN n is equal to, or leſs than -3, then the Value $\left(\frac{180}{\sqrt{n+3}}\right)$ of the Angle ACP, becoming either infinite or impoſſible, it follows, that if the Law of Centripetal Force be, as the Cube, or more than the Cube of the Diſtance inverſely, the Trajectory cannot have more than one *Apſide*: And, therefore, the Projectile in all ſuch Caſes muſt inevitably either fall into the Centre of Force, or fly from it *ad infinitum*, unleſs it moves in a Circle; which is agreeable to the *Scholium* aforegoing. But, if n be equal to 1, 0, — 1, or — 2; then will the Angular Diſtance of the two *Apſides* be, 90° : 00′, 103° : 55′, 127° : 17′, or 180° : 00′, reſpectively; the firſt and laſt of which we are aſſur'd of from other Principles.

COROL. II.

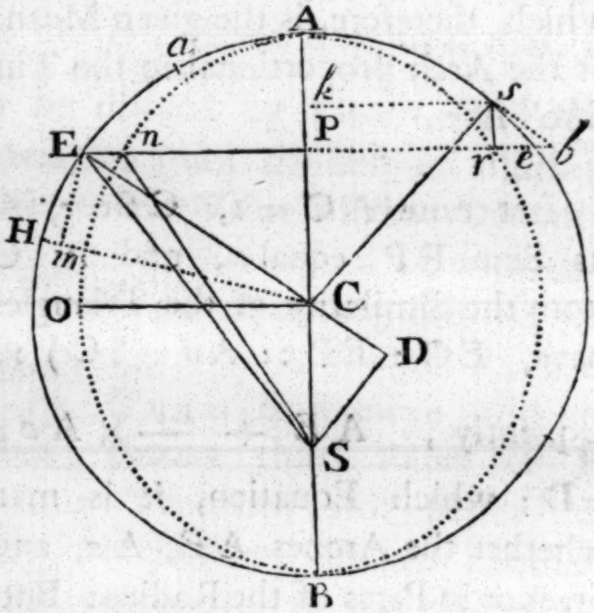
IF the Diſtance (D) of the *Apſides* be given, and the Law of Centripetal Force from thence be required: Then, by making $\frac{180}{\sqrt{n+3}}$ equal to D, we ſhall have $\overline{\frac{180}{D}}\Big|^2 - 3, = n$, for the Value ſought: Hence, if D be 360°, or the Body takes up one intire Revolution in going from one *Apſe* to the other; then, muſt the Law of Centripetal Force be reciprocally as that Power of the Diſtance, whoſe Exponent is $2\frac{1}{4}$; but, if either *Apſe*, from the Time of the Body leaving it, to its Return again, has mov'd forward only a very ſmall Diſtance, E, or D be $= 180° + \frac{E}{2}$, the Force will then be inverſely as the $2 + \frac{E}{180}$ Power of the Diſtance, very nearly.

SCHOLIUM.

IF x be any Diſtance of the Projectile from the Centre of Force, and the Law, by which it tends towards the ſame Centre, be every where, as $cx^n + dx^m + ex^p + fx^q$, &c. c, d, &c. n, m, &c. being determinate Quantities; and if a be the Diſtance of one of the *Apſides* from that Centre, the angular Diſtance of thoſe *Apſides* will be

$$\overline{\frac{ca^{n}+da^{m}+ea^{p}+fa^{q}\text{, \&c.}}{\overline{3+n}\times ca^{n}+\overline{3+m}\times da^{m}+\overline{3+p}\times ea^{p}}}\Bigg|^{\frac{1}{2}}\times 180^{\circ}.$$

From the MEAN ANOMALY *of a Planet given; to find its* PLACE *in its* ORBIT.



LET AOB be the given Orbit, S the Sun in one of the Foci, AC the Semi-Tranſverſe Axis, CO the Semi-Conjugate, AEHBA a Circle circumſcribing the Ellipſis, and let n be the Place of the Planet at any given Time after or before its paſſing, A, the Aphelion; thro' which draw EnP perpendicular to AB, and having joined the Points E S, E C, Sn, and made SD perpendicular to ECD, take the Arch EH equal to SD, and the Arch Aa equal to S C. Then, the Sector ECH being equal to the Triangle E C S, A C H A will be equal to A S E A; inaſmuch as the former of thoſe Areas is compounded of the Sector A C E and E C H, and the latter of the ſame Sector and the Triangle E C S: Wherefore, ſince the Area A S E A, is to A E B C A, half the Circle, as the Elliptical Area AnS A, to the Semi-Ellipſis AnB C A, by a known Relation of the two Curves; if, inſtead of A S E A, its Equal be ſubſtituted, we ſhall have, as A C H A : A E B A :: A *n* S A : A *n* B A; but A C H A is to A E B A, as the Arch A H to A E B, the Semi-Circumference; and therefore it will be, as A *n* B A : A *n* S A :: A E B : A H: Wherefore ſince the Areas A*n*BA, A*n*SA, deſcribed by Radii drawn to S, the Center of Force, are as the Times of their Deſcription, it will be, as the Time of deſcribing A *n* B A, or that of Half one Revolution, is to the given Time of deſcribing A *n* S A, ſo is A E B to A H; which, therefore, is the given Mean Anomaly in this Poſition, or the Arch proportional to the Time of the Planet's moving thro' A *n*.

Let now $AC = 1$, $CS = e$, $AH = D$, A E equal E, its Sine E P equal x, and its Co-ſine $CP = y$. Then, from the Similarity of the Triangles C E P, C S D, we ſhall have, EC : EP :: A*a* (SC) : EH (SD) and, conſequently, $AE + \frac{EP}{EC} \times Aa = AH$, or $E + x \times Aa$, $= D$; which Equation, it is manifeſt, will hold equally, whether the Arches A E, A *a*, and A H, be taken in Degrees or in Parts of the Radius: But now, in order to ſolve the ſame, let the required Arch, or Value of E, be eſtimated pretty near the Truth, and let this aſſumed Value be denoted by $As = \acute{E}$, its Difference (*se*) from the Truth, by $\dot{E}$; and $D - As - \frac{Rs}{AC} \times Aa = D - \acute{E} - \acute{x} \times Aa$, the Error of the the Equation, by $\acute{R}$; make *v r* parallel to A B, and let *s b* be a Tangent to the Circle at the Point *s*: Then, as *s b*, by reaſon of its Smallneſs, may, in this Caſe, be conſidered as equal to *e s*, and becauſe of the Similarity of the Triangles C*ks*, *srb*, we ſhall have as $1\ (Cs) : \acute{y}\ (Ck) :: \dot{E} : \acute{y} \times \dot{E} = rb$, or er, very nearly; whence $E = \acute{E} + \dot{E}$, and $PE = x = \acute{x} + \acute{y} \times \dot{E}$, which Values therefore being ſubſtituted in the general Equation $E + x \times Aa = D$, there comes out $\acute{E} + \dot{E} + \acute{x} \times Aa + \acute{y} \times \dot{E} \times Aa = D$ very nearly; wherefore $\dot{E} = \left(\frac{D - \acute{E} - \acute{x} \times Aa}{1 + \acute{y} \times Aa} =\right) \frac{D - \acute{E} - \acute{x} \times Aa}{1 + e\acute{y}}$ or, $= \frac{\acute{R}}{1 + e\acute{y}}$ nearly: Hence it appears, that, if the Error of the Equation be divided by $1 + e\acute{y}$, and the Quotient added to, or ſubtracted from the firſt or aſſumed Value of E, there will ariſe a new Value of that Quantity much nearer the Truth than the former: And if with this new Value, and thoſe of x and y correſponding thereto, we proceed to a new Error, or compute the Value of R, and that of the Diviſor $1 + ey$, &c. it is likewiſe evident, for the very ſame Reaſons, that a third Value of E may be found, by the ſame Theorem, ſtill nearer the Truth than the preceding, and from thence another, and ſo another, &c. 'till we arrive to any Accuracy deſired, each Operation, at leaſt, doubling the Number of Places; ſo that in the moſt excentric of the planetary Orbits two Operations will be found ſufficient to bring out the Angle ACE to leſs than a Second: And when that is known, as EP and SP are then given, the Angle nSP may be eaſily had; for, by the Property of Curve, it is $AC : CO :: EP : Pn = \frac{CO \times EP}{AC}$, and $SP : \frac{EP \times CO}{AC}\ (Pn) :: AC$ (Radius) : $\frac{EP \times CO}{SP} =$ the Tangent of ASn. *Q. E. I.*

Otherwiſe,

Otherwiſe,

Let Radius $EC = r$ and the general Equation $AE + \frac{EP}{EC} \times Aa = AH$, or $E = D - \frac{x}{r} \times Aa$ be again reſumed; then, the Orbit not being very Excentric, E will, it is evident, be nearly equal to D; and, conſequently, (x) the Sine of E, nearly equal to the Sine of D: Therefore, if the Sine of D be ſubſtituted for x, and the ſaid Sine be denoted by $\overset{\prime}{x}$ (ſignifying the firſt Value of x) it is obvious, that $D - \frac{\overset{\prime}{x}}{r} \times Aa$ will be nearer to the true Value of E, than D, and, conſequently, that the Sine of $D - \frac{\overset{\prime}{x}}{r} \times Aa$ (which I call $\overset{\prime\prime}{x}$) nearer to x than ($\overset{\prime}{x}$) the Sine of D; wherefore $D - \frac{\overset{\prime\prime}{x}}{r} \times Aa$, muſt be, ſtill, nearer the Truth, or the required Value of E, than $D - \frac{\overset{\prime}{x}}{r} \times Aa$, and, conſequently, its Sine (which I call $\overset{\prime\prime\prime}{x}$) ſtill, nearer x, than ($\overset{\prime\prime}{x}$) the Sine of $D - \frac{\overset{\prime}{x}}{r} \times Aa$: In like manner, the Sine of $D - \frac{\overset{\prime\prime\prime}{x}}{r} \times Aa$ (or $\overset{\prime\prime\prime\prime}{x}$) will appear to be nearer x than $\overset{\prime\prime\prime}{x}$, and $D - \frac{\overset{\prime\prime\prime\prime}{x}}{r} \times Aa$, nearer to the required Value than $D - \frac{\overset{\prime\prime\prime}{x}}{r} \times Aa$, *&c. &c.* Whence the following Method of Solution is manifeſt.

Let 1.758123, the Log. of (57.2958) the Number of Degrees in an Arch equal in Length to Radius, be added to the

Logarithm of the Excentricity, and from the Sum deduct the Logarithm of Half the greater Axis; the Remainder will be a 4th Logarithm (L); which, being once computed, will ſerve in all Caſes of that Orbit: To this Logarithm add the Logarithmical Sine of the given Mean Anomaly reckoned to or from the Aphelion; the Sum, rejecting Radius, will be the Logarithm of an Arch in Degrees; which, being taken from the Mean Anomaly, and the Sine of the Remainder added to the ſaid Logarithm, the Sum, rejecting Radius, will be the Logarithm of a 2d Arch; which, in like manner, being taken from the Mean Anomaly, and the Sine of the Remainder added to the ſame Logarithm, the Sum, rejecting Radius, will be the Logarithm of a 3d Arch; from whence, by repeating the Operation in the very ſame manner, a 4th Arch will be found, and ſo a 5th, &c. 'till we arrive to any aſſigned Exactneſs; the Error in the *Anomaly Excentri*, or Angle ACE, which Angle is to be expreſſed by the Difference of the Mean Anomaly and the laſt of the ſaid Arches, being always much leſs than the Difference of the ſaid Arch and that which immediately precedes it, from which Angle the true Anomaly is had as in the above Caſe. *Q. E. I.*

Otherwiſe,

The foregoing Conſtruction being retained, let Radius (AC) $=1$, the Sine of the given *Anomaly* ACH $=a$, its Co-ſine $=b$, and let Em be the Sine of EH: Then will $a\ \mathrm{C}m-b\times\mathrm{E}m=x$, the Sine of the Difference of thoſe Angles, by the Elements of *Trigonometry*; but EH being $=ex$, Em (by the ſame) will be $ex-\frac{e^3x^3}{2.3}+\frac{e^5x^5}{2.3.4.5}$, &c. and $\mathrm{C}m=1-\frac{e^2x^2}{2}+\frac{e^4x^4}{2.3.4}$, &c. whence, by Subſtitution, &c. we get $\overline{1+be}\times x$

$+\frac{ae^2x^2}{2}-\frac{be^3x^3}{6}-\frac{ae^4x^4}{24}$, *&c.* $=a$, where, by inverting the Series, x comes out $=\frac{a}{1+be}-\frac{\frac{1}{2}a^3e^2}{\overline{1+be}|^3}+\frac{a^3}{\overline{1+be}|^3}\times\overline{\frac{\frac{1}{6}be^3}{\overline{1+be}|^1}+\frac{\frac{1}{2}a^2e^3}{\overline{1+be}|^2}}$, *&c.* $=a\times\overline{1-be+\frac{2bb-aa}{2}\times ee+\frac{5aa-3bb}{3}\times be^3}$ *&c.* from whence, if the Excentricity be not very large, the Angle ASn may be had as above to any Degree of Exactneſs, *Q. E. I.*

Note, That, in theſe Solutions, when E is greater than a Right Angle, its Co-ſine y is to be conſidered as a negative Quantity.

SCHOLIUM.

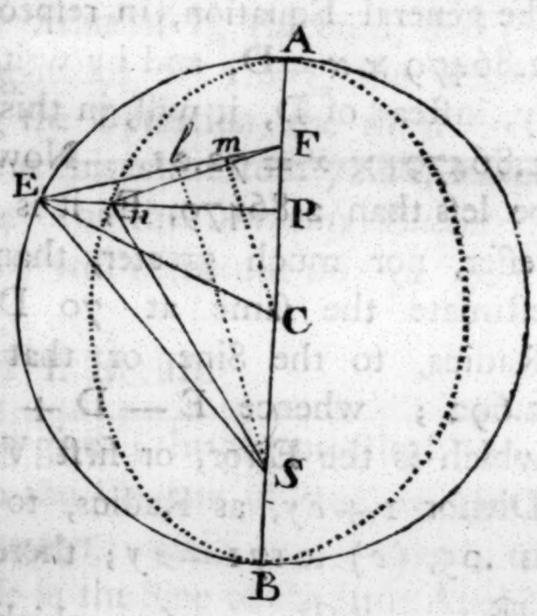
As the foregoing Methods of Solution may ſeem tedious or perplexed for common Practice, a ſhort Approximation, tho' limited in Point of Exactneſs, may be of Service. In order to this, we have ~~have~~ given EP $(=x)=a\times\overline{1-be+\frac{2bb-aa}{2}\times ee+\frac{5aa-3bb}{3}\times be^3}$, *&c.* as above, from whence PC $(=y=\sqrt{1-xx})$ is $=b+a^2e-\frac{3ba^2e^2}{2}+\frac{6bb-2aa}{3}\times a^2e^3+\frac{65a^2-6bb}{24}\times ba^2e^4$, *&c.* But, by the Nature of the Curve $1+ey$ is $=Sn$, and $Pn=\sqrt{1-ee}\times x\left(=\frac{OC\times EP}{AC}\right)$ and therefore $\frac{x\sqrt{1-ee}}{1+ey}\left(=\frac{AC\times Pn}{Sn}\right)$ will be the Sine of the Angle ASn, or true *Anomaly*, and $\frac{e+y}{1+ey}\left(=\frac{AC\times SP}{Sn}\right)$ its Co-ſine: Hence, by the Elements of Trigonometry $a\times\frac{e+y}{1+ey}-b\times\frac{x\sqrt{1-ee}}{1+ey}=\frac{ae+ay-bx\sqrt{1-ee}}{1+ey}$ will be the Sine of the Difference of the Angles HCA, ASn, or of the Equation of the Orbit; wherein, by ſubſtituting, inſtead of x and y, their reſpective Values, and contracting the whole by Diviſion, &c. there will come out $2ae$ into $1 - \frac{5be}{4} + \overline{\frac{3bb}{2} - \frac{4aa}{3}} \times ee + \overline{\frac{245aa}{48} - \frac{27bb}{16}} \times be^3$, &c. which, when e is not very large, will appear to be equal to $\frac{2ae}{1+\frac{5be}{4}} - \frac{1}{3} \times \overline{\frac{2ae}{1+\frac{5be}{4}}}\Big|^3$ very nearly; for this, converted to a Series, is $2ae - \frac{5abe^2}{2} + \frac{25ab^2e^3}{8} - \frac{8a^3e^3}{3} + 10a^3be^4 - \frac{125ab^3e^4}{32}$, &c. from which, if the former Series be taken, there will remain only $\frac{abbe^3}{8} + \overline{\frac{17bb}{32} - \frac{5aa}{24}} \times be^4$, &c. Hence is deduced the following

PRACTICAL RULE

For finding the Equation of the Centre from the Mean Anomaly given.

As Radius, to the Co-ſine of the given Anomaly, ſo is $\frac{5}{4}$ *Parts of the Excentricity of the Orbit, to a fourth Number; which Number add to half the greater Axis, if the Anomaly be leſs than* 90, *or more than* 270 *Degrees, otherwiſe ſubtract from the ſame: Say, as the Sum or Remainder, is to Double the Excentricity, ſo is the (Logarithmic) Sine of the given Anomaly, to the Sine of a firſt Arch; from three Times which Sine deduct the double Radius, the Remainder will be the Sine of a ſecond Arch, whoſe* $\frac{1}{3}$ *Part, taken from the former, leaves the Equation ſought.*

And it muſt be noted, that this *Rule*, in the Orbits of *Saturn*, *Jupiter*, and the *Moon*, anſwers to a Second, and in thoſe of the *Earth* and *Venus* to leſs than $\frac{1}{10}$ of a Second. And, in theſe two laſt, the Arch firſt found will, without farther Correction, be ſufficiently exact to anſwer to the niceſt Obſervations, the Error never amounting to above 2 or 3 Seconds; which is more correct than either the noted Hypotheſis of *Ward* or *Bullialdus*, as will appear from the following Examination of thoſe Hypotheſes, which, as they have been much celebrated, and come near the Truth in many Caſes, may here alſo deſerve a particular Conſideration. And, to begin with the latter, which ſuppoſes the Angle AFn made at F the upper Focus by the Aphelion and (n) the Planet to be the Mean Anomaly, and therefore SnF the Equation. Becauſe a the Sine and b the Co-ſine of the ſaid Angle are given, by the Nature of the Ellipſis, $Sn = \frac{1+2eb+ee}{1+be}$ is alſo given; whence, by *Plain Trigonometry*, it will be, as $\frac{1+2eb+ee}{1+be} : a :: 2e$ (SF) $: 2ae \times \frac{1+be}{1+2be+ee}$ equal to the Sine of SnF, which, put in a Series, is $2ae \times \overline{1 - eb + \overline{2bb - 1} \times ee + \overline{3 - 4bb} \times be^3}$, &c. and this taken from $2ae \times \overline{1 - \frac{5eb}{4} + \overline{\frac{3bb}{2} - \frac{4aa}{3}} \times ee}$, &c. leaves $2ae \times \overline{-\frac{eb}{4} + \overline{\frac{bb}{2} - \frac{aa}{3}}} \times ee$, &c. for the Error of this Hypotheſis. But now for the other, where, EnP being perpendicular to AB, AFE is ſuppoſed the Mean Anomaly. Let SC and Cm be perpendicular to FE; then it will be as 1 (EC) : a (the Sine of AFE, or CFE) :: e (CF) : ae=Cm, the Sine of CEF; whence Em, its Co-ſine, $=\sqrt{1-a^2e^2}$: Again, as 1 (the Sine of CmF) : e (CF) :: $-b$ (the Sine of FCm) : $-eb=$ Fm, equal alſo to Cm, becauſe CS is $=$ FC; for which Reaſon Sl is double to mC; wherefore it will be, as $\sqrt{1-a^2e^2}+eb$ (El) : $2ae$ (Sl) :: 1 (Radius) to $\frac{2ae}{\sqrt{1-a^2e^2}+eb}$ $=$ the Tangent of SEF; which, in a Series, will be $2ae\times\overline{1-eb+\frac{a^2e^2}{2}+e^2b^2}$, &c. whence the correſponding Sine is eaſily found $=2ae\times\overline{1-eb+b^2e^2-\frac{3aaee}{2}}$, &c. and this taken from $2ae\times\overline{1-\frac{5eb}{4}+\overline{\frac{3bb}{2}-\frac{4aa}{3}}\times e^2}$, gives $2ae$ into $-\frac{eb}{4}+\overline{\frac{bb}{2}+\frac{aa}{6}}\times ee$, &c. for the Error in this Caſe



Hence it will appear, that the greateſt Error of each of theſe Hypotheſes, in the Orbit of *Mars*, where e is upwards of .09, will be about 5 or 6 Minutes, and in the other planetary Orbits, according to the Squares of their Excentricities (in Parts of their own Semi-Axis) nearly; it alſo appears, that towards the Aphelion the Circular Hypotheſis will be the more correct, and near the Perihelion the other; and, laſtly, that both Hypotheſes make the Equation too large in the higher, and too ſmall in the lower, Part of the Orbit.

Having ſhewn how much theſe two noted Hypotheſes, (by many ſo much eſteemed) differ from Truth, it may be proper to proceed now to give ſome *Examples* of the preceding Methods, whereby the *Problem* is more correctly ſolved.

EXAMPLE I.

LET the Excentricity of the propoſed Orbit be $\frac{5}{100}$ of the Mean Diſtance or Semi-Tranſverſe Axis, and the given Mean Anomaly $72^\circ\ 12'\ 36'' = 72.21$ Degrees; and let the *Anomaly Excentri*, by the firſt Method, be required.

It will be, as 1, the Semi-Tranſverſe, to .05, the Excentricity, ſo is 57.2958, the Number of Deg. in an Arch, equal in Length to Radius, to 2.86479 = the Arch A *a*; wherefore, the general Equation, in reſpect to this Orbit, will be $E + 2.86479 \times x = D$, and by writing therein the given Anomaly, inſtead of D, it will, in this particular Caſe, become $E + 2.86479 \times x = 72.21$. Now, becauſe $2.86479 \times x$ muſt be leſs than 2.86479, E, it is evident, can neither be much leſſer, nor much greater, than 70 Degrees; therefore, I eſtimate the ſame at 70 Degrees, and then ſay, as Radius, to the Sine of that Angle, ſo is 2.86479, to 2.692; whence $E - D + 2.86479 \times x$ equal 0.482, which is the Error, or firſt Value of R: Again, for the Diviſor $1 + ey$, as Radius, to (y) the Co-ſine of 70°, ſo is .05, (e) $.0171 = ey$; therefore $1 + ey = 1.0171$, and $\frac{R}{1+ey} = \frac{0.482}{1.0171} = 0.464$; which, being taken from 70°, gives 69.536 for the next Value of E; wherefore, it will be, as Radius, to the Sine of 69.536, or $69^\circ\ 32'\frac{16}{100}$, ſo is the ſaid Co-efficient 2.86479, to 2.68401; from whence the next Value of R is found equal 0.01001; and this, divided by the next Value of $1+ey$, or even by 1.0171, the laſt Value, and the Quotient taken from 69.536, leaves the true Value of E (= 69° 31′ 34″) to leſs than a Second.

EXAMPLE II.

LET the ſame Things be propoſed, as in the preceding *Example*, and the Anſwer according to the ſecond Method be required.

The Value of L, or the Log. of the Arc A *a*, as found by the laſt Example, being 457093, I add thereto the Logarithmic Sine of 72° 12′ 36″, or 72° .21′, the Sum, rejecting Radius, is the Logarithm of 2.73, the firſt Arch, which ſubtracted from 72.21, the Remainder will be 69.48; to whoſe Sine adding the ſaid Value of L, the Sum, deducting Radius, will be the Logarithm of 2.683, the ſecond Arch; with which, repeating the Operation, the third Arch will come out 2.6838, *&c.* and this taken from 72.21, leaves 69.5261, *&c.* or 69° 31′ 34″ for the *Anomaly Excentri*; from whence the *True Anomaly* will come out 66° 52′ 50″.

EXAMPLE III.

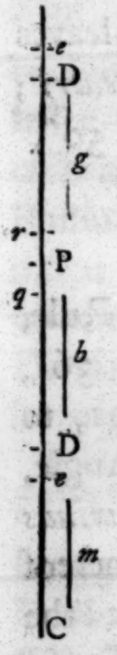
THE ſame Things being given; by the practical Rule, it will be, as Radius to the Co-ſine of 72° 12 36″, ſo is .0625 (=$\frac{5}{4}$ of .05) to .0191; again, as 1+.0191, to 0.1, the double Excentricity, ſo is the Sine of the ſame Angle, to the Sine of 5° 21′ 41″; three times whoſe Log. Sine, *minus* double the Radius, is the Sine of 2′ 48″; the $\frac{1}{3}$ Part whereof being taken from 5° 21′ 41″, leaves 5° 20′ 45″ for the Equation of the Center, and ∴ this taken from 72 12′ 36″ will give 66° 52′ 51″, equal to the *True* Anomaly very nearly.

Of

Of the Motion of Projectiles in reſiſting Mediums.

PROPOSITION I.

Suppoſing that a Body, let go from a given Point, with a given Velocity, directly to or from a Centre, towards which it uniformly gravitates, is reſiſted by a ſimilar Medium, in the Ratio of certain Powers of the Velocity, whoſe Indices are repreſented by the given Numbers, r, s, t, *&c. And ſuppoſing the Part of the whole Reſiſtance, at the ſaid given Point, correſponding to each of thoſe Powers, as well as the Force of Gravity, to be given; 'tis required to find the Relation of the Times, the Velocities, and the Spaces gone over.*



LET P be the given Point, DPC the Right Line in which the Body moves, and D, e, any two Points therein indefinitely near to each other: Suppoſe the Velocity at P to be ſufficient to carry the Body, uniformly, over a given Diſtance g, in a given Time h; and let m be the Space, which would be deſcribed in the ſame Time with the Velocity, that would be generated in that Time *in vacuo* by a Force equal to the Body's ſpecifick Gravity in the given Medium; let the Part of the Reſiſtance, which is as the r Power of the Celerity, at the aforeſaid Point, be ſuch, that the Body in moving over a given Diſtance b, with its Velocity uniformly continued, would from that Part alone, meet with a Reſiſtance ſufficient to take away its whole Motion; or which is the ſame, let b be the Diſtance that might be deſcribed with the Velocity at P in the Time that the Body would, by the ſaid Part alone, have all its Motion deſtroyed, was the Reſiſtance to continue the ſame as at the firſt Inſtant; and let the like Diſtances, with reſpect to the other Parts of the Reſiſtance, that are as the Powers of the Celerity, whoſe Indices are, s, t, *&c.* be c, d, *&c.* reſpectively; laſtly, let $PD = x$, $De = Pq = \dot{x}$, the Time of deſcribing $PD = T$, and the Space the Body would move over in the given Time b, with the Velocity at D, $= v$.

Then it will be, as $g : b :: \dot{x}\,(Pq) : \frac{b\dot{x}}{g}$ the Time of deſcribing Pq, and as $b ; g :: \dot{x}\,(Pq) : \frac{g\dot{x}}{b}$, the Velocity deſtroyed by that Part of the Reſiſtance, which is as the r Power of the Celerity, in that Time; therefore, the Velocity at D being to the Velocity at P, as v to g, that deſtroyed, by the ſame Part, in the ſame Time, from the Body's leaving D, will conſequently be $\frac{g\dot{x}}{b} \times \frac{v^r}{g^r} = \frac{v^r\dot{x}}{bg^{r-1}}$, becauſe this Part of Reſiſtance is as the r Power of the Velocity: But the Time of deſcribing De, is to the Time of deſcribing Pq, as g to v; therefore the Reſiſtance ariſing from the aforeſaid Part in deſcribing De, muſt be $\frac{v^r\dot{x}}{bg^{r-1}} \times \frac{b}{g} = \frac{v^{r-1}\dot{x}}{bg^{r-2}}$; from whence, it is manifeſt, by Inſpection, that the other Parts of the Reſiſtance, or Quantities of Motion deſtroyed thereby, will be $\frac{v^{s-1}\dot{x}}{cg^{s-2}}$, $\frac{v^{t-1}\dot{x}}{dg^{t-2}}$, *&c.* And therefore the whole Velocity deſtroyed by the Medium, in deſcribing De, is $\frac{v^{r-1}\dot{x}}{bg^{r-2}} + \frac{v^{s-1}\dot{x}}{cg^{s-2}}$, *&c.* But, the Time of deſcribing De, being to $\frac{b\dot{x}}{g}$ that of deſcribing Pq as g to v, will be repreſented by $\frac{b\dot{x}}{v}$; and therefore it will be, as $b : m :: \frac{b\dot{x}}{v} : \frac{m\dot{x}}{v}$, the Part of Velocity generated or deſtroyed in that Time, by the Force of Gravity, which added to, or taken from, the former Part, ariſing from the Reſiſtance, according as the Body is in its Aſcent or Deſcent, the Sum or Difference $\pm \frac{m\dot{x}}{v} + \frac{v^{r-1}\dot{x}}{bg^{r-2}} + \frac{v^{s-1}\dot{x}}{cg^{s-2}}$, &c. muſt, it is manifeſt, be equal to $(-\dot{v})$ the whole Decrement of Velocity: Hence we have $\dot{x} = \frac{-v\dot{v}}{\pm m + \frac{v^r}{bg^{r-2}} + \frac{v^s}{cg^{s-2}}, \text{ \&c.}}$.

Moreover, becauſe $\dot{T}$, the Time of deſcribing D e, is found equal to $\frac{b\dot{x}}{v}$, we have $\frac{\dot{T}v}{b} = \dot{x}$; which being ſubſtituted inſtead thereof in the other Equation, &c. there will come out $\dot{T} = \frac{-b\dot{v}}{\pm m + \frac{v^r}{bg^{r-2}} + \frac{v^s}{cg^{s-2}}, \text{ \&c.}}$. *Q. E. I.*

COROL. I.

HENCE, when the Reſiſtance is barely in the ſimple Ratio of the Velocity, then c, d, &c. being infinite, our Equations become $\dot{x} = \frac{-v\dot{v}}{\pm m + \frac{vg}{b}}$, and $\dot{T}$ equal $\frac{-b\dot{v}}{\pm m + \frac{vg}{b}}$: Whence $x = \frac{\overline{g-v}\times b}{g}$, $\pm \frac{mb^2}{g^2}$ into the Hyp. Log. of $\frac{gv \pm mb}{gg \pm mb}$, and $T = \frac{bb}{g}$ into Hyp. Log. of $\frac{gg \pm mb}{gv \pm mb}$.

COROL. II.

BUT, when the Reſiſtance is, as the Square of the Velocity, r being equal to 2, and c, d, &c. infinite (as before) the Equations will be $\dot{x}$ equal to $\frac{-v\dot{v}}{\pm m+\frac{v^2}{b}}$, and $\dot{T}$ equal to $\frac{-b\dot{v}}{\pm m+\frac{v^2}{b}}$: Hence x is found equal to $\frac{b}{2}\times$ Log. $\frac{g^2\pm mb}{v^2\pm mb}$, wherein, if v be taken $=0$, we ſhall have $\frac{b}{2}\times$ Hypb. Log. $1+\frac{g^2}{mb}$, for the Height of the whole Aſcent; but, if v^2-mb be taken $=0$, we ſhall have $\sqrt{mb}$ equal to the greateſt Velocity the Body can poſſibly acquire by deſcending; laſtly, if g be taken $=0$, there will be $\frac{b}{2}\times$ Log. $\frac{mb}{mb-vv}$ for the Diſtance gone over when the Body falls from Reſt; therefore, in that Caſe the Log. $\frac{mb}{mb-vv}$ being $=\frac{2x}{b}$, if n be put for the abſolute Number whoſe Hyperbolic Log. is $-\frac{2x}{b}$ we ſhall get $\frac{mb}{mb-vv}=\frac{1}{n}$, and conſequently $v=\overline{mb}|^{\frac{1}{2}}\times\overline{1-n}|^{\frac{1}{2}}$.

Moreover, with reſpect to the Time, becauſe $\dot{T}$ in the Deſcent of the Body is $=\frac{bb\dot{v}}{mb-vv}$ the Time it ſelf will, in this Caſe, be $\frac{b}{2}\sqrt{\frac{b}{m}}$ into the Hyp. Log. $\frac{\overline{mb}|^{\frac{1}{2}}+v}{\overline{mb}|^{\frac{1}{2}}-v}\times\frac{\overline{mb}|^{\frac{1}{2}}-g}{\overline{mb}|^{\frac{1}{2}}+g}$; and therefore, when the Body deſcends from Reſt, is barely $=\frac{b}{2}\sqrt{\frac{b}{m}}\times$ Log $\frac{\overline{mb}|^{\frac{1}{2}}+v}{\overline{mb}|^{\frac{1}{2}}-v}$, wherein, if the above found Value of v be ſubſtituted, it will be $\frac{b}{2}\sqrt{\frac{b}{m}}\times$

Log. $\frac{1+\overline{1-n}|^{\frac{1}{2}}}{1-\overline{1-n}|^{\frac{1}{2}}}$: But, in the other Caſe, $\dot{T}$ being $= \frac{-bb\dot{v}}{mb+vv}$ T will be equal to $\frac{b}{m}$ drawn into the Difference of the two Circular Arcs, whoſe Tangents are g and v, and whoſe common Radius is $\sqrt{mb}$. And, in like manner, the Values of x and T may be exhibited by the Quadratures, *&c.* of the Conic Sections, in any other Caſe, where the Reſiſtance is barely as a ſimple Power of the Velocity, whoſe Exponent is a rational Number, and alſo, in many Caſes, where the Reſiſtance is in the Ratio of two different Powers, by Help of the laſt *Problem* of this *Treatiſe.*

COROL. III.

IF m be taken $= 0$, or the Body be ſuppoſed to be affected by a Medium only, and the Reſiſtance be barely as a ſimple Power (r) of the Velocity; then $\dot{x}$ becoming equal $-bg^{r-2}v^{1-r}\dot{v}$, and $\dot{T} = -bbg^{r-2}v^{-r}\dot{v}$, x, in this Caſe, will therefore be $= \frac{b-bg^{r-2}v^{2-r}}{2-r}$, and T equal $\frac{bbg^{-1}-bbg^{r-2}v^{1-r}}{1-r}$: Where, if r be taken $= 0, 1, 2, 3,$ *&c.* ſucceſſively, x will be $\frac{b}{2} - \frac{bv^2}{2g^2}$, $b - \frac{bv}{g}$, b Log. $\frac{g}{v}$, $-b+\frac{bg}{v}$, *&c.* and T equal to $\frac{bb}{g} \times \overline{\frac{g-v}{g}}$, $\frac{bb}{g}$ Log. $\frac{g}{v}$, $\frac{bb}{g} \times \overline{\frac{g-v}{v}}$, $\frac{bb}{2g} \times \overline{\frac{gg-vv}{vv}}$, *&c.* reſpectively; from whence, by exterminating v, we have $x = \frac{gT}{b} \times \overline{1-\frac{gT}{2bb}}$, $T = \frac{bb}{g}$ Log. $\frac{b}{b-x}$, $x = b$ Log. $1+\frac{g}{b}\times\frac{T}{b}$, $T = b \times \frac{bx+\frac{1}{2}xx}{bg}$, *&c.* expreſſing the Relation of the Times and Spaces in the ſaid Caſes, reſpectively.

SCHO-

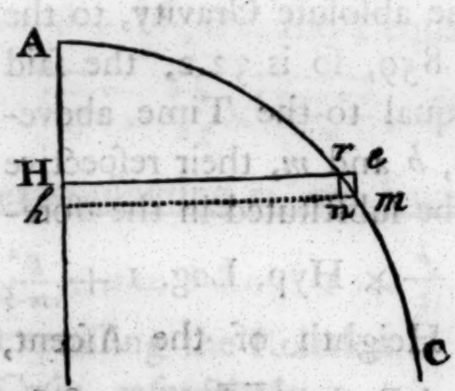
SCHOLIUM.

IN Fluids void of Tenacity the Reſiſtance is in the Duplicate Ratio of the Velocity; and it is found, that a Body in ſuch Fluids, by moving over a Space, which is to $\frac{8}{3}$ of its Diameter, as the Denſity of the Body, to that of the Medium, with its Velocity uniformly continued, would meet with a Reſiſtance ſufficient to take away its whole Motion: Therefore, if this Space be taken to repreſent the Value of b, in *Cor.* II. by Help of the *Theorems* there given, the Velocity, Time, or Space gone over, will be readily obtained. For an Inſtance hereof, let a Ball, whoſe Diameter is $\frac{1}{3}$ of a Foot, and whoſe Denſity is the ſame with that of common Rain-Water, be ſuppoſed to be projected upwards in a Direction perpendicular to the Horizon, with a Velocity ſufficient to carry it uniformly over a Space of 300 Feet in one Second of Time; and let the Heighth of the Aſcent, the Times of Aſcent and Deſcent, with the Velocity generated in Falling, be required. Becauſe, the Denſity of Rain-Water, is to that of Air, as 860 to 1, b will, here, be ($\frac{8}{3}\times\frac{1}{3}\times 860$) 764.4 Feet; and ſince the Velocity, which a Body would acquire in one Second of Time by freely deſcending *in vacuo*, is ſufficient to carry it uniformly over a Diſtance of 32.2 Feet in that Time, it will be, as the abſolute Gravity, to the ſpecifick Gravity; or, as 860, to 859, ſo is 32.2, the ſaid Feet, to ($32.16=$) m, h being equal to the Time abovementioned: Wherefore, if for g, h, b and m, their reſpective Values 300, 1, 764.4, and 32.16, be ſubſtituted in the aforeſaid *Theorems*, we ſhall have, firſt, $\frac{b}{2}\times$ Hyp. Log. $1+\frac{g^2}{mb}$ ($=x$) $=630$ Feet for the whole Heighth of the Aſcent, 2d, $\frac{b}{m}\times$ Arch, whoſe Tang. is g, and Rad $\sqrt{mb}$, $=5.48$ Seconds,

conds, the whole Time of Aſcent; 3^d, $n = .21455$; 4^{th}, $v = 139$, the Diſtance that would be uniformly deſcribed in one Second with the Velocity acquired by falling; laſtly, $\frac{b}{2}\sqrt{\frac{b}{m}} \times \text{Log.} \frac{1+\sqrt{1-n}}{1-\sqrt{1-n}} = 6.85$, the Time of Deſcent. But if the ſame Ball be ſuppoſed to move in Water with the ſame given Velocity; then, the ſpecifick Gravity in that Fluid being nothing, the Body may be conſidered as moving by its innate Force only; and, therefore, the Number of Feet gone over, in any Number of Seconds, denoted by T, will (by *Cor.* III.) be $\frac{8}{9}$, into the Hyp. Log. of $1 + 337.5\text{T}$.

PROPOSITION II.

To find the Reſiſtance and Denſity of a Medium, whereby a Body, gravitating uniformly in the Direction of Parallel Lines, is made to deſcribe a given Curve; the Law of Reſiſtance being given, partly as the n Power, partly as the 2 n Power, partly as the 3 n Power, &c. of the Celerity, or as $a\text{C}^n + b\text{C}^{2n} + c\text{C}^{3n}$, *&c. where* C *denotes the Celerity, and n, a, b, c, &c. any determinate Quantities.*



LET ArC be the propoſed Curve, and AH the Axis thereof, or a Right-line in which the Body gravitates, to which let rn and em be parallel, and Hr and hm perpendicular, r and m being any two Points in the Curve taken indefinitely near to one another: Suppoſe the Body arrived to r, with a Velocity in the Direction re, repreſented by v; let $AH = x$, $Hb\ (rn) = \dot{x}$, $Hr = y$, $nm\ (re) = \dot{y}$, $rm = z$, and let D be as the required Denſity. Then, ſince the Velocity in the Direction re is v, that in the Direction rn will be $\frac{v\dot{x}}{\dot{y}}$; and therefore $\frac{\dot{v}\dot{x}+v\ddot{x}}{\dot{y}}$ will be the Fluxion of the ſame, or the Increaſe of Velocity in that Direction during the Time of deſcribing rm; wherefore, if from this we take the Part ariſing from the Reſiſtance of the Medium, which is $\frac{\dot{v}\dot{x}}{\dot{y}}$ (becauſe it is to $\dot{v}$, the Alteration of Velocity in the Direction re, as $\dot{x}$ to $\dot{y}$) there will remain $\frac{v\ddot{x}}{\dot{y}}$ for the other Part ariſing from the Force of Gravity, in the ſame Time and Direction; therefore, the Reſiſtance in the ſaid Direction, is to the Force of Gravity, as $-\frac{\dot{v}\dot{x}}{\dot{y}}$ to $\frac{v\ddot{x}}{\dot{y}}$, or, as $-\frac{\dot{v}\dot{x}}{v\ddot{x}}$ to 1; and, conſequently, the abſolute Reſiſtance, in the Direction rm, to the Force of Gravity, as $\frac{-\dot{v}\dot{z}}{v\ddot{x}}$ to 1: But $\left(\frac{v\ddot{x}}{\dot{y}}\right)$ the Part of Velocity, ariſing from Gravity, being as the Time $\frac{\dot{y}}{v}$ of deſcribing rm, may be expreſſed thereby; whence we have $\frac{v\ddot{x}}{\dot{y}} = \frac{\dot{y}}{v}$, or $v^2\ddot{x} = \dot{y}\dot{y}$; and therefore in Fluxions $2v\dot{v}\ddot{x} + v^2\dddot{x} = 0$, or $-\frac{\dot{v}}{v} = \frac{\dddot{x}}{2\ddot{x}}$, which ſubſtituted in the foregoing Proportion $-\frac{\dot{v}\dot{z}}{v\ddot{x}} : 1$ gives $\frac{\dot{z}\dddot{x}}{2\ddot{x}\ddot{x}}$ to 1 for the Ratio of the Reſiſtance to the Gravity. Moreover, becauſe the abſolute Velocity is $\frac{v\dot{z}}{\dot{y}}$, the Reſiſtance, by Suppoſition, will be as D into $a\times\overline{\frac{v\dot{z}}{\dot{y}}}\Big|^{n}+b\times\overline{\frac{v\dot{z}}{\dot{y}}}\Big|^{2n}$ &c. or, becauſe $\frac{v\dot{z}}{\dot{y}}$ is $=\frac{\dot{z}}{\overline{\ddot{x}}\big|^{\frac{1}{2}}}$, as D into $a\times\frac{\dot{z}}{\ddot{x}^{\frac{n}{2}}}+b\times\frac{\dot{z}^{2n}}{\ddot{x}^{n}}$, &c. which Quantity muſt therefore be as $\frac{\dot{z}\dddot{x}}{\ddot{x}\ddot{x}}$, and conſequently D as

$$\frac{\dddot{x}}{a\dot{z}^{n-1}\times\ddot{x}^{\frac{4-n}{2}}+b\dot{z}^{2n-1}\ddot{x}^{\frac{4-2n}{2}}+c\dot{z}^{3n-1}\ddot{x}^{\frac{4-3n}{2}}\,\&c.}\quad Q.E.I.$$

COROLLARY.

HENCE, if the Law of Reſiſtance be only as a ſingle Power (n) of the Velocity; then, by taking b, c, d, &c. each = 0, and $a=1$, we have $\frac{\dddot{x}}{\dot{z}^{n-1}\times\ddot{x}^{\frac{4-n}{2}}}$ for the Denſity 'in that Caſe; which, therefore, when n is $=2$, or the Reſiſtance directly as the Square of the Velocity, will be barely as $\frac{\dddot{x}}{\dot{z}\ddot{x}}$.

EXAMPLE I

LET it be required to find the Denſity of a Medium, wherein a Body moving, ſhall deſcribe the common Parabola. Here x being $=\frac{y^{2}}{p}$, we have $\dot{x}$ equal

$\frac{2y\dot{y}}{p}$, $\ddot{x} = \frac{2\dot{y}\dot{y}}{p}$, and $\dddot{x} = 0$; and therefore $D = 0$; which ſhews, that a Body, to deſcribe this Curve, muſt move in Spaces entirely void of Reſiſtance.

EXAMPLE II.

TO find the Denſity, *&c.* when the Curve is a Circle, and the Reſiſtance as the Square of the Celerity.

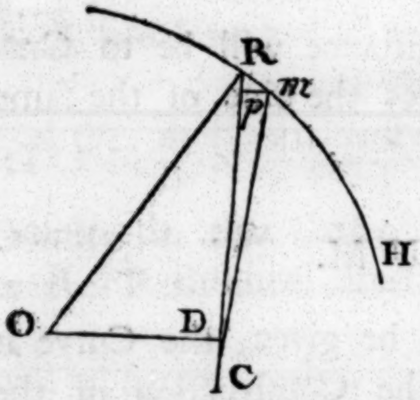
Becauſe x, in this Caſe, is $= a - \sqrt{aa - yy}$, there will be $\dot{x} = \frac{y\dot{y}}{\sqrt{aa-yy}}$, $\dot{z} = \frac{a\dot{y}}{\sqrt{aa-yy}}$, $\ddot{x} = \frac{aa\dot{y}\dot{y}}{\overline{aa-yy}|^{\frac{3}{2}}}$, and $\dddot{x} = \frac{3a^2y\dot{y}^3}{\overline{aa-yy}|^{\frac{5}{2}}}$; therefore the Denſity $\left(\frac{\dddot{x}}{\dot{z}\ddot{x}}\right)$ will here be, as $\frac{3y}{a\sqrt{aa-yy}}$, or, as the Tangent of the Diſtance from the higheſt Point directly; and the Reſiſtance will be to Gravity, as $3y$ to $2a$, or, as 3 Times the Sine of the ſame Diſtance to twice the Radius.

SCHOLIUM.

IF the Denſity of the Medium be given, the Curve it ſelf may be determined by the Conſtruction of the foregoing fluxional Equation: So, in caſe of an uniform Denſity, and a Reſiſtance, as the Square of the Velocity, where we have $D = \frac{\dddot{x}}{\dot{z}\ddot{x}}$, or $D\ddot{x}\sqrt{\dot{y}\dot{y} + \dot{x}\dot{x}} = \dddot{x}$; x will be found $= \frac{y^2}{2p} + \frac{Dy^3}{3p} + \frac{D^2y^4}{12p}$, *&c.* And when D is conſtant, and the Reſiſtance barely as the Velocity, it will be $x = -\frac{4}{DD} \times \text{Log.}\ \overline{1 - \frac{Dy}{2p}}|^{\frac{1}{2}}, -\frac{4y}{\overline{2p}|^{\frac{1}{2}} \times D}$; p, in either Caſe, being twice the Radius of Curvature, or the Parameter at the Vertex; both which, and the true Value of D, may be eaſily computed from the Velocity at A, and the given Denſity of the Medium.

PROPOSITION III.

The Centripetal Force being given, and the Law of Reſiſtance, as any Power (n) of the Velocity; to find the Denſity of a Medium in each Part thereof, whereby a Body may deſcribe a given Spiral about the Centre of Force.



LET RmH be the given Spiral, R and m two Points therein as near as may be to each other, and C the Centre of Force; and let RO be the Radius of Curvature at R; making OD and mp perpendicular to RC, and calling RD, s; RC, x; Rp, $\dot{x}$; Rm, $\dot{z}$; the Centripetal Force, C; and the Velocity, v. Foraſmuch, as RO is to RD, as the abſolute Centripetal Force at R, to that, which tending to the Centre O, would be ſufficient to retain the Body in the Circle, whoſe Radius is RO, and, becauſe the Centripetal Forces in Circles, the Velocities being the ſame, are inverſely as the Radii; RD, it is manifeſt, is the Radius of the Circle which might be deſcribed with the Velocity and Centripetal Force at R: Therefore, ſince the Centripetal Force, in Circles, is known to be ſuch, as is ſufficient to generate or deſtroy all the Velocity of the moving Body, in the Time it is uniformly deſcribing a Diſtance equal to the Semi-diameter of its Orbit, we have, $s\ (RD) : \dot{z}\ (Rm) :: v : \left(\frac{v\dot{z}}{s}\right)$ the Velocity which the Centripetal Force would generate in another Body freely deſcending from Reſt at R, in the Time the former is deſcribing R m; wherefore, by the Reſolution of Forces, it will be $\dot{z}\ (Rm) : \dot{x}\ (Rp) :: \frac{v\dot{z}}{s} : \frac{v\dot{x}}{s}$, the Velocity generated in the ſame Time, by the Body deſcribing Rm; which, therefore, added to $\dot{v}$ the Exceſs of the Velocity at R above that at m, the Sum $\frac{v\dot{x}}{s} + \dot{v}$ will, it is manifeſt, expreſs the Velocity taken away by the Medium in that ſame Time: But the Velocities generated or deſtroyed in equal Bodies, in equal Times, are as the Forces by which they are generated or deſtroyed; and, therefore, it will be as $\frac{v\dot{x}}{s} + \dot{v} : \frac{v\dot{z}}{s}$, or as $\frac{\dot{x}}{\dot{z}} + \frac{s}{\dot{z}} \times \frac{\dot{v}}{v}$ to 1, ſo is the Reſiſtance to the Centripetal Force. But, the Velocities in Circles being in the ſubduplicate Ratio of the Radii and Centripetal Forces conjunctly, v will be as $\sqrt{sC}$, and conſequently $\frac{\dot{v}}{v} = \frac{\dot{s}}{2s} + \frac{\dot{C}}{2C}$; whence, by Subſtitution, it will be as $\frac{2\dot{x} + \dot{s}}{2\dot{z}} + \frac{s\dot{C}}{2C\dot{z}} : 1$, or, as $\overline{2\dot{x} + \dot{s}} \times \frac{C}{2\dot{z}} + \frac{s\dot{C}}{2\dot{z}} : C$, ſo is the Reſiſtance to the Centripetal Force; but C is the Centripetal Force, and therefore $\overline{2\dot{x} + \dot{s}} \times \frac{C}{2\dot{z}} + \frac{s\dot{C}}{2\dot{z}}$ is the required Reſiſtance; which being divided by $(\overline{sC}|^{\frac{n}{2}})$

the *n* Power of the Velocity, becauſe the Reſiſtance is in the Ratio thereof, and the Denſity of the Medium conjunctly, the Quotient $\frac{2\dot{x}+\dot{s}}{2s^{\frac{n}{2}}\dot{z}C^{\frac{n-2}{2}}} + \frac{\dot{C}}{2C^{\frac{n}{2}}\dot{z}s^{\frac{n-2}{2}}}$ will, it is manifeſt, be as the Denſity of the Medium. *Q. E. I.*

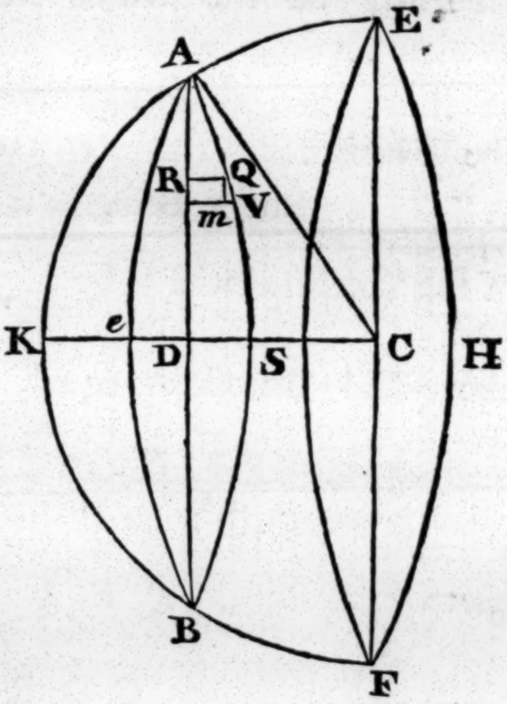
EXAMPLE.

LET the Reſiſtance be in the duplicate Ratio of the Velocity, the Centripetal Force as ſome Power, *m*, of the Diſtance, and the Curve propoſed the Logarithmic Spiral; and, Radius being *r*, let *c* be the Co-ſine of the common Angle, which all the Ordinates make with the Spiral: Then *s*, by the Nature of the Curve, being $= x$ ($= CR$) $\frac{2\dot{x}+\dot{s}}{2\dot{z}}$ will be $= \frac{3\dot{x}}{2\dot{z}} = \frac{3c}{r}$, and therefore $\frac{2\dot{x}+\dot{s}}{2\dot{z}} + \frac{s\dot{C}}{2C\dot{z}} = \frac{3c}{2r} + \frac{m\dot{x}}{2\dot{z}} = \frac{c}{r} \times \frac{m+2}{2}$; hence we have, as $\frac{m+3}{2} : \frac{r}{c}$, ſo is the Reſiſtance to the Centripetal Force. And the Denſity of the Medium will be as $\frac{\overline{c\times m+3}}{2rx}$; that is, when *c* and *m* are given, reciprocally as the Diſtances from the Centre of Force: But when *m* is -3, then $\frac{\overline{c\times m+3}}{2rx}$ becoming $=0$, it appears, that the Body in this Caſe muſt move in Spaces entirely void of Reſiſtance to deſcribe the propoſed Spiral: And, therefore, the Law of Centripetal Force being more than the Cube of the Diſtance inverſely, the Deſcription of this Curve will, it is manifeſt, be impoſſible from any reſiſting Force whatſoever.

Of the Motion and Reſiſtance of Pendulous Bodies in a Medium.

PROPOSITION I.

Suppoſing two equal Pendulums, whoſe Bobs are in Form of the Segments of Spheres, to be moving with equal Velocities in a reſiſting Medium, and the Thickneſs of each Bob with the Diameter of the Sphere from which it is formed, to be given; To find the Ratio of their Reſiſtances.



LET AKBA be one Side, or Half, of one of the propoſed Bobs, and EAKBFCE Half the whole Sphere whereof it is a Segment; and let the ſaid Segment be conceived to be divided into an indefinite Number of indefinitely ſmall *Laminæ*, by circular Planes perpendicular to the Axis KC, and equi-diſtant from each other; and let AeBSA be one of thoſe *Laminæ*, included between any two adjacent Planes, and n be the Thickneſs thereof, or the common Diſtance of the ſaid Planes; calling AC, a; AD, c; any Ordinate RQ, y; mv, $\dot{y}$; and KD, x. Then, by the Property of the Circle, we have DR $= \sqrt{cc - yy}$; which, Radius being a, is the Sine of the Angle that the Surface at Q makes with *m* Q, the Direction of its Motion, or the Incidence of the Particles it ſtrikes againſt: Therefore, ſince the reſiſting Force of the Medium, on any Surface the Velocity being the ſame, is as the Number of Particles falling thereon, and the Square of the Sine of their common Incidence conjunctly, the whole Reſiſtance of that Part of the propoſed Surface, repreſented by Q*v*, will be as $\frac{cc-yy}{aa}\times n\dot{y}$; becauſe $n\dot{y}$ is evidently as the Number of Particles: Hence, by taking the Fluent, we have $\frac{ny}{aa}\times\overline{cc-\tfrac{1}{3}y^2}$ for the Reſiſtance of A Q; the Double whereof, $\frac{4c^3n}{3aa}$, when y becomes equal c, will conſequently be the whole Reſiſtance of the ſaid *Lamina*: Which Reſiſtance, if the Axis KD (x) be, now, ſuppoſed to flow, and $\dot{x}$ be put inſtead of n, will, it is manifeſt, be the Fluxion of the Reſiſtance of the propoſed Segment AKBD: But x being $=a-\sqrt{aa-cc}$ by the Property of the Circle $\dot{x}$ will ∴ be $=\frac{c\dot{c}}{\sqrt{aa-cc}}$, and conſequently the above-ſaid Fluxion equal to $\frac{4c^4\dot{c}}{3aa\sqrt{aa-cc}}$; whoſe Fluent will be $\frac{AK\times CK}{2}-\frac{AD\times DC}{2}\times\overline{1+\frac{2AD^2}{3CK^2}}$, and therefore the Double thereof as the whole Reſiſtance of the given Pendulum. *Q. E. I.*

COROL. I.

HENCE it appears, that the Reſiſtance of the whole generating Sphere will be expreſs'd by EK × KC, or the Area of the Semi-circle EKFE; and therefore is to that of its circumſcribing Cylinder, moving in the Direction of its Axis, exactly, as 1 to 2.

COROL. II.

IF the Reſiſtance, as above found, be divided by 3.14159, *&c.* $\times KD^2 \times \frac{6KC - 2KD}{3}$, the ſolid Content, or Quantity of Matter in the Pendulum, the Quotient will, it is manifeſt, be as the Retardation of its Velocity ariſing from that Reſiſtance; and this, if b be put for the Axis or Thickneſs of the Bob, and d its greateſt Diameter, will be equal to $\frac{3bbd}{\overline{bb+dd}|^2}$ very nearly.

COROL. III.

WHEREFORE, if b be taken $= d$, we ſhall have $\frac{3}{4d}$ for the Retardation of the Globe, whoſe Diameter is d; and therefore the Retardation of the Pendulum to that of its circumſcribing Sphere will be as $\frac{3bbd}{\overline{bb+dd}|^2}$ to $\frac{3}{4d}$, or as $\overline{2bd}^2 : \overline{bb+dd}|^2$ nearly.

Note, If the Bobs of Pendulums be in other Forms than thoſe of Segments of Spheres, the Reſiſtance will be readily had as above; ſince it is evident, that A C (a) being taken for the Normal of the generating Curve K A, the Reſiſtance of the *Lamina* A*e*BSA will be $\frac{4c^3n}{3aa}$ as is there found, let that Curve be what it will.

LEMMA.

The Reſiſtance of a Body in a Medium, is to the Force of Gravity, as twice the Space thro' which the Body muſt freely fall by that Gravity to acquire the given Velocity, to the Space

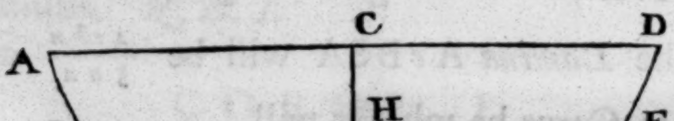
Space over which it might move with that Velocity in the Time wherein the ſaid Reſiſtance, uniformly continued, would take away the Body's whole Motion.

For the Velocity acquired by freely deſcending from Reſt, thro' any Space, is known to be generated in the Time that the Body, with the Velocity ſo acquired, would move uniformly over double that Space: But the Forces, by which the ſame Motion would be uniformly generated or deſtroyed, are inverſely as the Times in which it might be generated or deſtroyed; and therefore inverſely, as the Diſtances deſcribed with the ſame Velocity in thoſe Times.

PROPOSITION II.

Suppoſing that a heavy pendulous Body, oſcillating in a Cycloid, is reſiſted by an uniform Force, and at the ſame time by a rare and ſimilar Medium, in the duplicate Ratio of the Velocity; To find the Exceſs of the Arc, deſcribed in the whole Deſcent above the Arc, deſcribed in the ſubſequent Aſcent, and the Time of one entire Oſcillation.

LET ABD be the whole Cycloid, BC its Axis, EB the Arc deſcribed in the Deſcent, and BF that deſcribed in the ſubſequent Aſcent; draw GHE, F*f*, &c. parallel to AD, let S be any Place of the Body, and *h* R, the



the Diſtance thro' which it muſt freely deſcend in *vacuo*, by a Force equal to its ſpecifick Gravity in the given Medium, to acquire the ſame Velocity as it has in that Place. Suppoſe that Part of the Reſiſtance, which is uniform, to be to the Force of Gravity, as m to 1; and let d be the Space over which the Body muſt move with its Velocity uniformly continued, to meet with a Reſiſtance from the other Part, ſufficient to take away its whole Motion; or, which is to the ſame Effect, let d be to the indefinitely ſmall Arc Sn, as the whole Motion of the Body at S, to that deſtroyed by the Medium in moving thro' Sn; calling BD, b; BE, a; Rb, z; ES, x; Sn, $\dot{x}$. Now, the Force of Gravity being repreſented by 1, that Part of it whereby the Body is accelerated, at the Point S, is $\frac{a-x}{b}$ $\left(=\frac{BS}{BD}\right)$: And, by the preceding *Lemma*, we have, as $d : 2z :: 1 : \frac{2z}{d}$ for the latter Part of the Reſiſtance, or that in the duplicate Ratio of the Velocity; which being added to m, the former Part, and the whole taken from $\frac{a-x}{b}$, gives $\frac{a-x}{b} - m - \frac{2z}{d}$ for the whole Force whereby the Body is accelerated at the ſaid Point: Therefore the Velocity, there, being known to be, as $\sqrt{2z}$, that generated in $\frac{\dot{x}}{\sqrt{2z}}$ the Time of deſcribing Sn will be defined by $\overline{\frac{a-x}{b} - m - \frac{2z}{d}} \times \frac{\dot{x}}{\sqrt{2z}}$; which muſt therefore be equal to, $\frac{\dot{z}}{\sqrt{2z}}$, the Fluxion or Increaſe of, $\sqrt{2z}$, the aforeſaid Velocity: Hence, we have $\frac{a-x}{b} - m - \frac{2z}{d} = \frac{\dot{z}}{\dot{x}}$; which, by writing e inſtead of $a - mb$, *&c.* becomes $-\frac{e}{b}+\frac{x}{b}+\frac{2z}{d}+\frac{\dot{z}}{\dot{x}}$ equal 0; from whence, by ſolving the Equation, z is found $= \frac{ex}{b} - \overline{\frac{a+2e}{2ba}\times x^2 - \frac{2x^3}{3d} + \frac{4x^4}{3\cdot4a^2} - \frac{8x^5}{3\cdot4\cdot5d^3}}$, *&c.* Or, if p be put for (0.367878) the Number, whoſe hyperbolical Logarithm is -1, $= \overline{1-p^{\frac{2x}{d}}} \times \frac{\overline{\frac{1}{2}dd+de}}{2b} - \frac{dx}{2b}$. But when the Body arrives at F, the Hight of its Aſcent, z, or its Equal $\frac{ex}{b} - \overline{\frac{d+2e}{2bd}\times x^2 - \frac{2x^3}{3d} + \frac{4x^4}{3\cdot4d^2}}$, *&c.* becomes equal 0; which Equation ſolved, gives $x = 2e - \frac{4e^2}{3d} + \frac{16e^3}{9d^2}$, *&c.* $=$ EBF; therefore FG is $= 2a - 2e + \frac{4e^2}{3d} - \frac{16e^3}{9dd}$, *&c.* $= 2mb + \frac{4e^2}{3d} - \frac{16e^3}{9dd}$, *&c.* (by reſuming $2mb$ inſtead of its Equal $2a - 2e$;) which, becauſe m and $\frac{e}{d}$ are ſuppoſed very ſmall, will be $2mb + \frac{4a^2}{3d}$ very nearly. *Q. E. I.*

Moreover, ſince the Time of deſcribing the leaſt poſſible Diſtance Sn, is as $\frac{\dot{x}}{\sqrt{2z}}$, by ſubſtituting therein the Value of z, as above found, we ſhall have

$$\frac{\dot{x}}{\overline{\frac{2ex}{b} - \overline{\frac{d+2e}{bd}}\times x^2 - \overline{\frac{2x^3}{3d} + \frac{4x^4}{3\cdot4\cdot d^2}},\ \&c.}\Big|^{\frac{1}{2}}} = \frac{b^{\frac{1}{2}}}{\overline{1+\frac{2e}{d}}\Big|^{\frac{1}{2}}\times x^{\frac{1}{2}}}$$

$$\times \frac{\dot{x}}{\overline{\frac{2e}{1+\frac{2e}{d}} - x + \frac{2x^2}{3d} - \frac{4x^3}{3\cdot4d^2},\ \&c.}\Big|^{\frac{1}{2}}}$$ for the Fluxion of the required Time: But, becauſe $\frac{2e}{1+\frac{2e}{d}} - x + \frac{2x^2}{3d}$, *&c.* the Square of the Diviſor of the latter Part or Factor thereof, when x becomes

becomes $= 2e - \frac{4ee}{3d} + \frac{16e^3}{9d^2}$, &c. appears, from above, to be equal to Nothing, if r be put to denote the Value of $2e - \frac{4ee}{3d} + \frac{16e^3}{9dd}$, &c. or the Root of the Equation, it is manifeſt that $\frac{2e}{1+\frac{2e}{d}} - r + \frac{2r^2}{3d} - \frac{4r^3}{3.4dd}$ will alſo be equal 0, and conſequently $r - \frac{2r2}{3d} + \frac{4r^3}{3.4dd}$, &c. $= \frac{2e}{1+\frac{2e}{d}}$; which being ſubſtituted inſtead thereof, our ſaid Fluxion will become $\frac{b^{\frac{1}{2}}}{\overline{1+\frac{2e}{d}}|^{\frac{1}{2}} \times x^{\frac{1}{2}}} \times \frac{\dot{x}}{\overline{r - \frac{2r^2}{3d} + \frac{4r^3}{3.4dd}, \text{\&c.} - x + \frac{2x^2}{3d}, \text{\&c.}}|^{\frac{1}{2}}}$

$= \frac{b^{\frac{1}{2}}}{\overline{1+\frac{2e}{d}}|^{\frac{1}{2}} \times x^{\frac{1}{2}}} \times \frac{\dot{x}}{\overline{r - x \text{ in } 1 - 2 \times \frac{r+x}{3d} + 4 \times \frac{r.r + rx + xx}{3.4dd}, \text{\&c.}}|^{\frac{1}{2}}} =$

$\frac{b^{\frac{1}{2}} x^{-\frac{1}{2}} \dot{x}}{\overline{1+\frac{2e}{d}}|^{\frac{1}{2}} \times \overline{r-x}|^{\frac{1}{2}}}$ into $\overline{1 - 2 \times \frac{r+x}{3d} + 4 \times \frac{rr + rx + xx}{3.4dd}, \text{\&c.}}|^{-\frac{1}{2}}$,

and laſtly, by converting $\overline{1 - 2 \times \frac{r+x}{3d}, \text{\&c.}}|^{-\frac{1}{2}}$ into a rational Series, equal to $\frac{b^{\frac{1}{2}} x^{-\frac{1}{2}} \dot{x}}{\overline{1+\frac{2e}{d}}|^{\frac{1}{2}} \times \overline{r-x}|^{\frac{1}{2}}}$ into $\overline{1 + \frac{r+x}{3d} + \frac{rx}{6dd}}$ &c. Now, the Fluent of $\frac{x^{-\frac{1}{2}} \dot{x}}{\overline{r-x}|^{\frac{1}{2}}}$ $\left(= \frac{\dot{x}}{\sqrt{rx - xx}}\right)$ when x is $= r$, is known to be equal to the Periphery of the Circle whoſe Diameter is Unity; wherefore, if that Periphery be put equal p, the required Fluent of our given Expreſſion $\frac{b^{\frac{1}{2}} x^{-\frac{1}{2}} \dot{x}}{\overline{1+\frac{2e}{d}}|^{\frac{1}{2}} \times \overline{r-x}|^{\frac{1}{2}}} \times \overline{1 + \frac{r+x}{3d} + \frac{rx}{6dd}}$, &c. will then appear to be $\frac{b^{\frac{1}{2}}p}{\overline{1+\frac{2e}{d}}\big|^{\frac{1}{2}}} \times \overline{1+\frac{r+\frac{1}{2}r}{3d}+\frac{rr}{12dd}}$, &c. from *p.* 118. of my Book of *Fluxions*; which, by reſtoring the known Value of r, will become $pb^{\frac{1}{2}} \times \overline{1+\frac{2e}{d}}\big|^{-\frac{1}{2}} \times \overline{1+\frac{e}{d}-\frac{e^2}{3dd}}$, &c. $= pb^{\frac{1}{2}} \times \overline{1-\frac{e}{d}+\frac{3e^2}{2dd}-\frac{5e^3}{2d^3}}$, &c. $\times \overline{1+\frac{e}{d}-\frac{e^2}{3dd}+\frac{4e^3}{9d^3}}$ &c. $= pb^{\frac{1}{2}} \times \overline{1+\frac{e^2}{6dd}-\frac{2e^3}{9d^3}}$, &c. and this is, it is manifeſt, as the Time of one entire Oſcillation. *Q. E. I.*

COROL. I.

WHEN d is infinite, then FG becoming barely equal $2mb$, it will be as $2b : FG :: 1 : m$: Hence it appears, that the Exceſs of the Arc deſcribed in the whole Deſcent above that deſcribed in the ſubſequent Aſcent, when the Reſiſtance is uniform, is to twice the Length of the Pendulum, or DBA, as the reſiſting Force, to the Force of Gravity.

COROL. II.

BUT, when m is $= 0$, or the Reſiſtance barely in the duplicate Ratio of the Velocity; the ſaid Exceſs will be in the duplicate Ratio of the Velocity, or Arc deſcribed nearly.

COROL. III.

IF m be conſidered as negative, or the Pendulum, inſtead of being reſiſted by an uniform Force, be accelerated thereby, ſo as to continue its Vibrations in the ſame given

Arc; then, ſince $2mb + \frac{4aa}{3d}$ (= FG) becomes = o, $-m$ will be $= \frac{2aa}{3bd}$: And, therefore, it is manifeſt, that the Force, which acting uniformly on the given Body, is ſufficient to counter-ballance a Reſiſtance in the duplicate Ratio of the Velocity, or to keep it vibrating in the ſame given Arc, muſt be to the Weight of the Pendulum, as $2aa : 3bd$ nearly: And, therefore, the Arcs, which a given Pendulum ſo actuated, will continue to deſcribe, by different Forces, will be nearly as the Square Roots of thoſe Forces.

COROL. IV.

WHEN both m and $\frac{a}{d}$ are equal to nothing, that is, when the Oſcillations are performed without Reſiſtance, the Time of Vibration will be barely $pb^{\frac{1}{2}}$; which is to the Time, wherein a Body freely deſcending from Reſt, would fall thro' CB, Half the Length of the Pendulum, as the Circumference of a Circle, to its Diameter.

COROL. V.

MOREOVER, when only $\frac{a}{d}$ is = o, or the Reſiſtance uniform, the Vibrations will, alſo, be Iſochronal, and performed in the very ſame Time as if the Pendulum was not at all reſiſted.

COROL. VI.

BUT if m be equal to nothing, or the Pendulum be reſiſted, only, in the duplicate Ratio of the Velocity; the Time of Oſcillation will then be $pb^{\frac{1}{2}} \times$

$\overline{1+\frac{a^2}{6dd}-\frac{2a^3}{9d^3}}$, &c. Therefore, the Exceſs of the Time of one whole Vibration, in a Medium reſiſting in the duplicate Ratio of the Velocity, above the Time of Vibration in the leaſt Arc poſſible, is to the Time of Vibration in this Arc, as $\frac{aa}{6ad}-\frac{2a^3}{9ddd}$, &c. to Unity; or, becauſe $\frac{a}{d}$ is very ſmall, as $\frac{aa}{6dd}$, to 1 very nearly: Hence it ſhould follow, that the ſaid Exceſs, is in the duplicate Ratio of the Arcs very nearly, I ſay *ſhould follow*, becauſe I know very well, that Sir *Iſaac Newton*, Princip. Prop. 27. B. II. makes it to be, nearly, in the ſimple Ratio of the Arcs: This I confeſs had made me more than a little ſuſpect, that I might have here fallen into an Error; and yet upon re-examining the Proceſs with more than ordinary Attention, I have not been able to diſcover any Miſtake therein committed; but, if any ſuch ſhould occur to my Readers, I ſhall readily acknowledge my ſelf obliged for the Diſcovery.

SCHOLIUM.

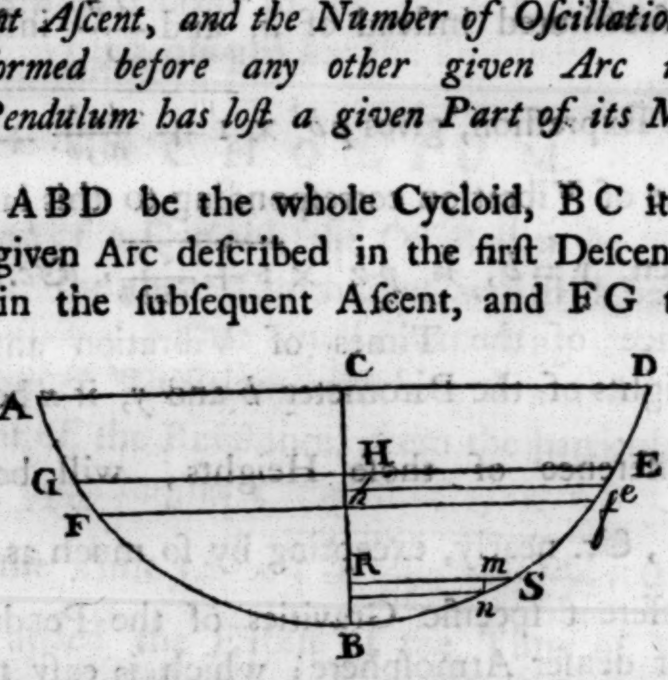
IF inſtead of a Cycloid, the Oſcillations be performed in a Circle, the above Concluſions will ſtill hold, provided the Arc deſcribed be but ſmall; excepting thoſe that relate to the Time of Vibration, which is ſhortned or prolonged, independent of the Reſiſtance, from the particular Nature of the Curve, according as a ſmaller or greater Arc is deſcribed. But, if to the Time $pb^{\frac{1}{2}}\times\overline{1+\frac{a^2}{6d^2}-\frac{2a^3}{9d^3}}$, &c. found as above, be added the Exceſs of the Time of Vibration in the Arch *a*, of a Circle whoſe Radius is *b*, above the Time, in the leaſt Arc poſſible, which, by *p.* 140. of my Book of

Fluxions, is $pb^{\frac{1}{2}}\times\overline{\frac{aa}{16bb}}$, &c. we ſhall then have $pb^{\frac{1}{2}}\times\overline{1+\frac{a^2}{16bb}+\frac{a^2}{6d^2}}$, &c. for the Time of Oſcillation in the Arc a of that Circle nearly. Hence it will not be difficult to determine, how much the Times of Vibration, in ſmall Arcs of Circles, are increaſed or decreaſed from the different Weights of the Atmoſphere. For, if the Force by which the Pendulum is kept in Motion, be always the ſame, the Arc deſcribed, by *Cor.* III. will be as $\sqrt{bd}$, that is, in the ſubduplicate Ratio, inverſely, of the Denſity of the Medium, or Height of the Barometer: Therefore, if b be put for the Height of the Barometer, at the Time when a given Arc c is deſcribed, the Length of the Vibration correſponding to (y) any other Height thereof, will be $\frac{cb^{\frac{1}{2}}}{y^{\frac{1}{2}}}$; this therefore being ſubſtituted inſtead of a, and $\frac{db}{y}$ inſtead of d, in the above Expreſſion, gives $pb^{\frac{1}{2}}\times\overline{1+\frac{bc^2}{16yb^2}+\frac{yc^2}{6bdd}}$, &c. for the Time of Vibration correſponding to this laſt Height; which, when $y=b$, is $pb^{\frac{1}{2}}\times\overline{1+\frac{c^2}{16bb}}$, &c. therefore the Difference of the Times of Vibration anſwering to the two Heights of the Barometer b and y, if n be put equal to the Difference of thoſe Heights, will be $\frac{pb^{\frac{1}{2}}n}{y}\times\overline{\frac{c^2}{16bb}-\frac{yc^2}{6bdd}}$, &c. nearly, excepting by ſo much as it is varied thro' the different ſpecific Gravities of the Pendulum, &c. in a rarer or denſer Atmoſphere; which is eaſy to be computed. However, after all, it is not to be ſuppoſed, that the Alteration in the Time of Vibration, above ſpecified, will happen immediately upon the Riſe or Fall of the Mercury; becauſe the Pendulum, thro' its *vis inertiæ*, will be ſome Time before it can be brought to perform its Vibrations, either in a greater or ſmaller Arc: And, indeed, the Alterations, both in the Time and Arc deſcribed, from the above Cauſe, are ſo ſmall, when compared with thoſe ariſing from Friction and Expanſion, as ſcarcely to come under the niceſt Obſervation.

PROPOSITION III.

Suppoſing that a heavy Body, oſcillating in a Cycloid, is reſiſted by a rare and ſimilar Medium in the Ratio of a given Power of (n) of the Velocity; to find the Exceſs of the Arc deſcribed in the whole Deſcent above that deſcribed in the ſubſequent Aſcent, and the Number of Oſcillations that will be performed before any other given Arc is deſcribed, or the Pendulum has loſt a given Part of its Motion.

LET ABD be the whole Cycloid, BC its Axis, EB the given Arc deſcribed in the firſt Deſcent, BF that deſcribed in the ſubſequent Aſcent, and FG the required Difference of thoſe Arcs; and, ſuppoſing the Body to be arrived at any Point S, let its Velocity there be the ſame as it



it would acquire in freely deſcending from Reſt thro' the Arc eS by an uniform Gravity equal to its ſpecifick Gravity in the given Medium; and let d be to the Length of the Arc BD, as the ſaid Gravity to the Reſiſtance, which the Body would ſuffer with the Velocity that it might acquire, from that Gravity, by freely falling thro' CB: Draw EHG, SR, *&c.* parallel to AD, and let b, a, $A^{\frac{1}{2}}$, and z, ſtand for BD, BE, Be, and BS reſpectively. Therefore, the Velocity acquired *in vacuo*, being in the ſubduplicate Ratio of the Diſtance perpendicularly deſcended, $\overline{Rb}|^{\frac{1}{2}}$, or its Equal $\sqrt{A - zz}$ (from the Property of the Curve) will be as the Velocity at S; and therefore $\frac{-\frac{1}{2}\dot{A} + z\dot{z}}{\sqrt{A - zz}}$ the Fluxion thereof, as the Increaſe of that Velocity: But, this Increaſe depends upon two Cauſes; the one, the Force of Gravity, and the other, the Reſiſtance of the Medium. If the Medium did not reſiſt, A would be conſtant, and therefore the Increaſe of the Velocity barely as $\frac{\dot{z}z}{\sqrt{A - zz}}$; wherefore, the other Part, ariſing from the Reſiſtance, muſt be as $\frac{-\frac{1}{2}\dot{A}}{\sqrt{A - zz}}$, and, conſequently, the reſiſting Force of the Medium, to that Part of the Gravity by which the Body is accelerated, as $\frac{\frac{1}{2}\dot{A}}{\sqrt{A - zz}} : \frac{z\dot{z}}{\sqrt{A - zz}}$, or, as $\dot{A}$ to $2z\dot{z}$: But this Part of the Gravity being to the whole, as z to b, by the Property of the Curve; the Reſiſtance will, it is manifeſt, be to the ſpecific Gravity, as $\dot{A}$ to $2b\dot{z}$. Moreover, becauſe the Velocity, at the ſame Point S, is to the Velocity which would be acquired by freely deſcending (as above) along BC, as $\sqrt{A - zz} : b$; the Reſiſtance, with the former of thoſe Velocities, will be to the Reſiſtance with the

latter,

latter, as $\overline{A - zz}|^{\frac{n}{2}}$ to b^{n}, or, as $\frac{\overline{A - zz}|^{\frac{n}{2}}}{b^{n-1}}$ to b; and, conſequently, the Reſiſtance at S, to the Force of Gravity, as $\frac{\overline{A - zz}|^{\frac{n}{2}}}{b^{n-1}}$ to d: Wherefore, it will be as $\frac{\overline{A - zz}|^{\frac{n}{2}}}{b^{n-1}} : d ::$ $\dot{A} : 2b\dot{z}$; whence $\frac{\dot{A}}{2} = \frac{\dot{z} \times \overline{A - zz}|^{n}}{db^{n-2}}$, from which Equation A may be determined by the known Methods of infinite Series, &c. be the Medium what it will: But, in a very rare one, ſuch as is ſuppoſed in the *Propoſition*, the Thing may be, otherwiſe, much more eaſily effected. For, then Ee, being, at its greateſt, exceeding ſmall, $aa - zz$, may, without ſenſible Error, be ſubſtituted in our Equation for $A - zz$; which done, we have $\frac{\dot{A}}{2}$ equal $\dot{z} \times \frac{\overline{aa - zz}|^{\frac{n}{2}}}{db^{n-2}}$: And then the Fluent of the latter Part thereof, when z is equal a, and n an even Number, will be found to come out $\frac{a^{n+1}}{db^{n-2}}$ into $\frac{2}{3} \times \frac{4}{5} \times \frac{6}{7} \times \frac{8}{9}$, &c. to $\frac{n}{2}$ Factors; and when z equal a, and n an odd Number, equal to $\frac{3.14159. \&c. \times a^{n+1}}{2db^{n-2}}$ in $\frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \frac{7}{8}$, &c. to $\frac{n+1}{2}$ Factors; from which two Expreſſions, the Fluent of the ſaid Part, in any intermediate Caſe, where n is a Fraction, may, by Interpolation, be nearly obtained: But, $\frac{A}{2}$ the Fluent of the contrary Side, when S coincides with B, is $= \frac{Be^{2}}{2}$, and when S and e coincide with E, equal

$\frac{BE^2}{2}$; wherefore, if s be put for the Uncia of the Fluent, above found, we ſhall have $\frac{sa^{n+1}}{db^{n-2}}$ equal $\frac{BE^2 - Be^2}{2}$ equal $\frac{\overline{BE+Be}\times\overline{BE-Be}}{2} = \frac{\overline{BE+Be}\times Ee}{2}$; whence, becauſe B$e$ is nearly equal to B E, we get E$e = \frac{sa^n}{db^{n-2}}$, that is in the aforeſaid Circumſtance, when S coincides with B; therefore $\frac{2sa^n}{db^{n-2}}$, the Double thereof, will, it is evident, be equal to F G very nearly. *Q. E. I.*

Let x, now, be the Number of Vibrations performed before the Pendulum is brought to deſcribe, in its whole Deſcent, any ſmaller Arc (E): Then, ſince ($\dot{E}$) the Decrement of this Arc, in one entire Oſcillation, or while x is increaſed by 1, or $\dot{x}$, is found to be $= \frac{2sE^n}{db^{n-2}}$, we have $\dot{E} = \frac{2sE^n}{db^{n-2}}$, or $\frac{2sE^n\dot{x}}{db^{n-2}}$, and therefore $\dot{x} = \frac{db^{n-2}E}{2sE^n}$; whence $x = \frac{db^{n-2}}{2s} \times \frac{E^{1-n} - a^{1-n}}{n-1}$. *Q. E. I.*

COROL. I.

THE Difference of the two Arcs deſcribed, in the Deſcent and ſubſequent Aſcent, is in the Ratio of the ſame Power of either of thoſe Arcs, as the Reſiſtance is of the Velocity.

COROL. II.

THE Reſiſtance being in the duplicate Ratio of the Celerity, and the Lengths of the two Arcs given; the Number of Oſcillations betwixt the Times of deſcribing thoſe Arcs will continue the ſame very nearly, let the Cycloid, or Length of the Pendulum, be what it will.

COROL. III.

IF the Reſiſtance be either uniform, or directly in the Ratio of any Power of the Velocity leſs than in the ſimple Ratio, as the ſubduplicate, ſubtriplicate, *&c.* the Body will continue vibrating 'till it hath compleated $\frac{d\,a^{1-n}}{2sb^{2-n}\times\overline{1-n}}$ entire Oſcillations, and then will have entirely loſt all its Motion.

COROL. IV.

WHEN the Reſiſtance is in more than the ſimple Ratio of the Velocity, the Motion will be prolonged *ad infinitum*.

COROL. V.

LASTLY, if any two Arcs of the Cycloid, or ſmall Arcs of a Circle, be taken in a given Ratio to each other, the Number of Vibrations performed between the Times of deſcribing thoſe Arcs, in one whole Deſcent of the Pendulum, will be nearly in the inverſe Ratio of that Power of either of the ſaid Arcs, whoſe Exponent is leſs by Unity than that expreſſing the Ratio of the Reſiſtance;

that is, if two Arcs, A, B, be taken in the ſame Ratio, as two other Arcs, C, D, the Number of Vibrations betwixt deſcribing the two former, will be to the Number betwixt deſcribing the two latter, in one whole Deſcent of the Pendulum, as C^{n-1} to A^{n-1}, or as D^{n-1} to B^{n-1}. From whence, and the foregoing Concluſions, not only the Law, but the abſolute Reſiſtance of Mediums may be found, by obſerving the Number of Vibrations performed therein by given Pendulums, in loſing given Parts of their Motion.

A new Method for the Solution of Equations in Numbers.

CASE I.

When only one Equation is given, and one Quantity (x) to be determined.

TAKE the Fluxion of the given Equation (be it what it will) ſuppoſing, x, the unknown, to be the variable Quantity; and having divided the whole by $\dot{x}$, let the Quotient be repreſented by A. Eſtimate the Value of x pretty near the Truth, ſubſtituting the ſame in the Equation, as alſo in the Value of A, and let the Error, or reſulting Number in the former, be divided by this numerical Value of A, and the Quotient be ſubtracted from the ſaid former Value of x; and from thence will ariſe a new Value of that Quantity much nearer to the Truth than the former, wherewith proceeding as before, another new Value may be had, and ſo another, *&c.* 'till we arrive to any Degree of Accuracy deſired.

CASE II.

When there are two Equations given, and as many Quantities (x and y) to be determined.

TAKE the Fluxions of both the Equations, conſidering x and y as variable, and in the former collect all the Terms, affected with $\dot{x}$, under their proper Signs, and having divided by $\dot{x}$, put the Quotient $= A$; and let the remaining Terms, divided by $\dot{y}$, be repreſented by B: In like manner, having divided the Terms in the latter, affected with $\dot{x}$, by $\dot{x}$, let the Quotient be put $= a$, and the reſt, divided by $\dot{y}$, $= b$. Aſſume the Values of x and y pretty near the Truth, and ſubſtitute in both the Equations, marking the Error in each, and let theſe Errors, whether poſitive or negative, be ſignified by R and r reſpectively: Subſtitute likewiſe in the Values of A, B, a, b, and let $\frac{Br - bR}{Ab - aB}$ and $\frac{aR - Ar}{Ab - aB}$ be converted into Numbers, and reſpectively added to the former Values of x and y; and thereby new Values of thoſe Quantities will be obtained; from whence, by repeating the Operation, the true Values may be approximated *ad libitum*.

Note, 1. That every Equation is firſt to be ſo reduced by Tranſpoſition, that the Whole may be equal to Nothing.

2. That, if after the firſt Operation, the Value of x or y be not found to come out pretty nearly as aſſumed, ſuch Value is not to be depended on, but a new Eſtimation made, and the Operation begun again.

3. That, the above Method, for the general part, when x and y are near the Truth, doubles the Number of Places at each Operation, and only converges ſlowly, when the Diviſor A, $Ab - aB$, at the ſame time converges to nothing.

EXAMPLE I.

LET $300x - x^3 - 1000$ be given $= 0$; to find a Value of x. From $300\dot{x} - 3x^2\dot{x}$, the Fluxion of the given Equation, having expunged $\dot{x}$, (*Caſe* I.) there will be $300 - 3xx = A$: And, becauſe it appears by Inſpection, that the Quantity $300x - x^3$, when x is $= 3$, will be leſs, and when $x = 4$, greater than 1000, I eſtimate x at 3.5, and ſubſtitute inſtead thereof, both in the Equation and in the Value of A, finding the Error in the former $= 7.125$, and the Value of the latter $= 263.25$: Wherefore, by taking $\frac{7.125}{263.25} = .027$ from 3.5 there will remain 3.473 for a new Value of x; with which proceeding as before, the next Error, and the next Value of A, will come out .00962518, and 263.815 reſpectively; and from thence the third Value of $x = 3.47296351$; which is true, at leaſt, to 7 or 8 Places.

EXAMPLE II.

LET $\sqrt{1 - x} + \sqrt{1 - 2xx} + \sqrt{1 - 3x^3} - 2 = 0$. This in Fluxions will be $\frac{-\dot{x}}{2\sqrt{1 - x}} - \frac{2x\dot{x}}{\sqrt{1 - 2xx}} - \frac{9x^2\dot{x}}{2\sqrt{1 - 3x^3}}$, and therefore A, here, $= -\frac{1}{2\sqrt{1 - x}} - \frac{2x}{\sqrt{1 - 2xx}} \frac{-9x^2}{2\sqrt{1 - 3x^3}}$; wherefore if x be ſuppoſed $= .5$, it will become -3.545: And, by ſubſtituting 0.5 inſtead of x in the given Equation, the Error will be found .204; therefore $\frac{.204}{-3.545}$ (equal — .057) ſubtracted from .5, gives .557 for the next Value of x; from whence, by proceeding as before, the next following will be found .5516, &c.

EXAMPLE III.

LET there be given the Equations $y+\sqrt{y^2-x^2}-10=0$, and $x+\sqrt{yy+x}-12=0$; to find x and y.

The Fluxions here being $\dot{y}+\frac{y\dot{y}-x\dot{x}}{\sqrt{yy-xx}}$ and $\dot{x}+\frac{y\dot{y}+\frac{1}{2}\dot{x}}{\sqrt{yy+x}}$ or $\dot{y}+\frac{y\dot{y}}{\sqrt{yy-xx}}-\frac{x\dot{x}}{\sqrt{yy-xx}}$, and $\dot{x}+\frac{\frac{1}{2}\dot{x}}{\sqrt{yy+x}}+\frac{y\dot{y}}{\sqrt{yy+x}}$, we have A equal $-\frac{x}{\sqrt{yy-xx}}$, B equal $1+\frac{y}{\sqrt{yy+xx}}$, $a=1+\frac{\frac{1}{2}}{\sqrt{yy+x}}$, and $b=\frac{y}{\sqrt{yy+x}}$ (*Caſe* II.)

Let x be ſuppoſed equal 5, and y equal 6; then will R equal — .68, r equal — .6, A equal — 1.5, B equal 2.8, a equal 1.1, b equal 9; therefore $\frac{Br-bR}{Ab-aB}=.23$, and $\frac{aR-Ar}{Ab-aB}=.37$, and the new Values of x and y equal to 5.23, and 6.37 reſpectively; which are as near the Truth as can be exhibited in three Places only, the next Values coming out 5.23263 and 6.36898.

Note, When Equations are given to be ſolved in this manner, it will be convenient, that they be firſt of all reduced to the moſt commodious Forms, to facilitate the Operations, whether into Fractions or Surds, or *vice verſa*: For Inſtance, the Equations in the laſt Example had been much eaſier ſolved, had they been firſt reduced, out of Surds, to $20y - xx - 100 = 0$, and $yy - xx + 25x - 144$ equal 0, or, by exterminating y, and working according to *Caſe* I. whereas, on the other hand, to have reduced the Equation, in the preceding Example, out of Surds (as is uſual in other Methods) would have rendered the Trouble of Solution almoſt inſuperable.

EXAMPLE. IV.

LET $49 \times \overline{x - \frac{x}{x+y|^{2}}} - 25 \times \overline{1 - \frac{xx}{1+y|^{2}}} = 0$; and $81 \times \overline{1 - \frac{xx}{1+y|^{2}}} - 49 \times \overline{1 \frac{x}{y} - \frac{xy}{1+x|^{2}}} = 0$.

Here, taking the Fluxions of both the Equations, and proceeding according to *Caſe* II. we have A equal $49 \times \overline{1 + \frac{x-y}{x+y|^{3}}} + \frac{50x}{1+y|^{2}}$, $\text{B} = \frac{98x}{x+y|^{3}} - \frac{50x^{2}}{1+y|^{3}}$, $a = \frac{-162x}{1+y|^{2}} + 49 \times \overline{\frac{y}{1+x} - \frac{1}{y} - \frac{2xy}{1+x|^{3}}}$, and $b = \frac{162x^{2}}{1+y|^{3}} + 49x \times \overline{\frac{1}{yy} + \frac{1}{1+x|^{2}}}$.

Suppoſe $x = .8$, and $y = .6$; then will be found $\text{R} = .45$, $r = 2.66$, $\text{A} = 68$, $\text{B} = 20.7$, $a = -131$, $b = 146$, and the next Values of x and y equal to .799 and .582; with which, repeating the Operation, the next following will come out .79912 and .58138, both which are true, at leaſt, to 4 Places: But, if a greater Exactneſs ſhould be deſired, let the Operation be once more repeated, and then the next Values will be true to double thoſe Places.

N. B. Altho' in ſeveral Caſes it happens, that the required Values, from the Equations themſelves, cannot be aſſumed near the Truth without ſome Attention and Trouble; yet, from the Nature of the *Problem* from whence thoſe Equations are derived, when that is known, the Trouble may be avoided, and the Thing effected without any great Difficulty: For inſtance, tho' it is not eaſy to perceive, that y and x are about $\frac{6}{10}$ and $\frac{8}{10}$ in the laſt Example; yet, when it is known, that 1, x, and y, are the Sides of a Plain Triangle, wherein Lines, drawn to biſect each Angle and terminate in thoſe Sides, are to one another, reſpectively, as 5, 7, and 9, the Thing then appears evident upon the firſt Conſideration.

EXAMPLE V

LET $x^x + y^y - 1000 = 0$, and $x^y + y^x - 100 = 0$. Here we ſhall have $A = \overline{1 + L : x} \times x^x$, B equal $\overline{1 + L : y} \times y^y$, $a = \frac{y}{x} \times x^y + y^x L : y$, and b equal $\frac{x}{y} \times y^x + x^y L : x$. Now, it appearing from the firſt Equation, that the greateſt of the two required Quantities cannot be leſſer than 4, nor greater than 5; and from the firſt and ſecond together, that the Difference of x and y muſt be pretty large; otherwiſe $x^x + y^y$ could not be 10 times as great as $x^y + y^x$: I therefore take x (which I ſuppoſe the greater Number) equal 4.5, and y equal 2.5; and then by a Table of Logarithms, or otherwiſe, find the next Values of theſe Quantities to be 4.55 and 2.45; and the next following 4.5519, &c. and 2.4495, &c. reſpectively.

Of INCREMENTS.

PROPOSITION I.

If, $\underset{''}{n}$, $\underset{'}{n}$, n, n', n'', n''', &c. *be a Series of Terms in a decreaſing Arithmetical Progreſſion, whoſe common Difference is* $\dot{n}$; *and* $n'\,n''\,n'''\ldots\ldots\times\overset{\overset{r}{\smile}}{n}$, *a Product ariſing from the Multiplication of any Number*, r, *of thoſe Terms, immediately ſucceeding each other, continually together; and if each of the Factors in this Product be increaſed by the common Difference: I ſay, the Product it ſelf will be increaſed by* $r\,\dot{n}\times n'\,n''\,n'''\ldots\ldots\times\overset{\overset{r-1}{\smile}}{n}$; *taken under the firſt Values of thoſe Quantities.*

FOR, ſince n', increaſed by the common Difference, becomes equal to n; and n'', increaſed by the ſame Difference, equal to n', *&c. &c.* it is manifeſt, that the new Value of the ſaid Product, ariſing from ſuch an Increaſe of its Factors, will be equal to $n\,n'\,n''\ldots\ldots\times\overset{\overset{r-1}{\smile}}{n}$, under the firſt Values of thoſe Quantities, from which taking the former, or given Value of that Product, we have $n\,n'\,n''\ldots\times\overset{\overset{r-1}{\smile}}{n} - n'\,n''\,n'''\ldots\times\overset{\overset{r}{\smile}}{n} = n'\,n''\,n'''\ldots\times\overset{\overset{r-1}{\smile}}{n}$ into $n-\overset{\overset{r}{\smile}}{n}$ for the Increment; which, becauſe the Exceſs of n above $\overset{\overset{r}{\smile}}{n}$ is equal to r times the common Difference, will conſequently be equal to $r\,\dot{n}\times n'\,n''\,n'''\ldots\ldots\overset{\overset{r-1}{\smile}}{n}$. *Q. E. D.*

COROL-

COROLLARY.

SINCE the Increment of $\overset{\prime}{n}\overset{\prime\prime}{n}\overset{\prime\prime\prime}{n}\ldots\ldots\times\overset{\underset{\smile}{r}}{n}$ is proved to be $= r\dot{n}\times\overset{\prime}{n}\overset{\prime\prime}{n}\overset{\prime\prime\prime}{n}\ldots\ldots\times\overset{\underset{\smile}{r-1}}{n}$, that of $\dfrac{\overset{\prime}{n}\overset{\prime\prime}{n}\overset{\prime\prime\prime}{n}\ldots\ldots\times\overset{\underset{\smile}{r}}{n}}{r\dot{n}}$ muſt conſequently be $\overset{\prime}{n}\overset{\prime\prime}{n}\overset{\prime\prime\prime}{n}\ldots\ldots\times\overset{\underset{\smile}{r-1}}{n}$: Whence, to find a Product of this kind from its Increment given, the following Rule is derived, *i. e.* Increaſe the Number of Factors, by annexing to them the next inferior Term of the Progreſſion, and divide the whole by the Number of Factors, thus increaſed, drawn into the common Difference.

Note, That $\overset{\underset{\smile}{r-1}}{n}$ ſtands for the Term of the propoſed Progreſſion, whoſe Diſtance from n is $r-1$, when on the deſcending Side, and $\underset{\overset{\frown}{r-1}}{n}$ when on the aſcending Side; the like is to be underſtood of any other.

EXAMPLE I.

LET a Product or Quantity, expreſſing the Value of $1+2+3+4+5+\ldots\ldots n$ be required; or, which is the ſame in effect, let it be required to find an Expreſſion ſo affected, that increaſing n by 1 (or writing therein $n+1$ inſtead of n) it ſhall be augmented by $n+1$. Then, if $\underset{\prime}{n}, n, \overset{\prime}{n}$, &c. be aſſumed for a Series of Numbers in Arithmetical Progreſſion, whoſe common Difference is 1, according to the above Notation, we ſhall have $\underset{\prime}{n}$ equal to the given Increment in this Caſe; to which annexing n, the next inferior Term of the Progreſſion, and dividing the Product $(\underset{\prime}{n}n)$ by (2) the Number of Factors, drawn into the common Difference; there comes out $\frac{1}{2}\underset{\prime}{n}n$, or $\frac{\overline{n+1}\times n}{2}$, which is equal $1+2+3+4+5\ldots+n$, the Value propoſed.

EXAMPLE II.

LET it be required to find the Sum of a Series of Cubes, as $1+8+27+64+125$, &c. Put n for the Number of Cubes to be taken, S their required Sum, and $\dot{S}$ the next ſucceeding Cube of the Series after the laſt in S, or the Increment of S that will ariſe by augmenting (n) the Number of Terms by Unity: Then, becauſe 1 is equal to the Root of the firſt Term, and alſo equal to the common Difference, the Root of the Term $\dot{S}$, it is manifeſt, will be $= n+1$; or, if $\underset{\prime\prime}{n}$, $\underset{\prime}{n}$, n, $\overset{\prime}{n}$, &c. be put for a Series of Numbers whoſe common Difference is 1, equal to n; wherefore $\dot{S}$ is equal $\underset{\prime}{n}\underset{\prime}{n}\underset{\prime}{n}$. But to bring this Value of $\dot{S}$ to the Form of the Propoſition; inſtead thereof, let its Equal $\overline{\underset{\prime\prime}{n}-1}\times\underset{\prime}{n}\times\overline{n+1}$, or $\underset{\prime\prime}{n}\underset{\prime}{n}n+\underset{\prime}{n}$ be ſubſtituted; then S, according to the Rule, will be $\frac{\underset{\prime\prime}{n}\underset{\prime}{n}n\overset{\prime}{n}}{4}+\frac{\underset{\prime}{n}n}{2}$ where, for $\underset{\prime\prime}{n}\overset{\prime}{n}$, writing its Equal $n\underset{\prime}{n}-2$, it will become $S=\frac{\overline{\underset{\prime}{n}n}\,|^{2}}{4}=\frac{\overline{n+1\times n}\,|^{2}}{4}$

EXAMPLE III.

TO find the Unciæ of a Binomial raiſed to a given Power.

Let $a+b$ be the propoſed Binomial, n the Exponent of its Power, and let B, C, &c. be the required Unciæ, or, which is the ſame in effect, let $\overline{a+b}|^{n}$ be $= a^{n} + \text{B}\,a^{n-1} \times b + \text{C}a^{n-2}\,b^{2} + \text{D}\,a^{n-3}\,b^{3}$, &c.

The Equation multiplied by $a+b$ becomes $\overline{a+b}|^{n+1} = a^{n+1} + \left.\begin{matrix}\text{B}\\1\end{matrix}\right\}a^{n}b + \left.\begin{matrix}\text{C}\\\text{B}\end{matrix}\right\}a^{n-1}b^{2} + \left.\begin{matrix}\text{D}\\\text{C}\end{matrix}\right\}a^{n-2}b^{3} + \left.\begin{matrix}\text{E}\\\text{D}\end{matrix}\right\}a^{n-3}b^{4}$ &c. wherefore, becauſe the Unciæ of the Power, whoſe Exponent is $\left\{\begin{matrix}n\\n+1\end{matrix}\right\}$ are $\left\{\begin{matrix}1, & \text{B}, & \text{C}, & \text{D}, & \text{E}, & \text{F}, & \text{G}, & \&c.\\1, & \text{B}+1, & \text{C}+\text{B}, & \text{D}+\text{C}, & \text{E}+\text{D}, & \text{F}+\text{E}, & \text{G}+\text{F}, & \&c.\end{matrix}\right.$ it is manifeſt, that the Values of B, C, D, &c. are ſuch, that increaſing (n) the Exponent by Unity, they will be increaſed by 1, B, C, &c. reſpectively. But the Increment of B being $= 1$, the Value of B, or Increment of C, will be $= n$; and therefore C, or the Increment of D, equal to $\frac{n\overset{\prime}{n}}{2}$; therefore D equal to $\frac{n\overset{\prime}{n}\overset{\prime\prime}{n}}{2.3}$; therefore F equal to $\frac{n\overset{\prime}{n}\overset{\prime\prime}{n}\overset{\prime\prime\prime}{n}}{2.3.4}$ &c. &c. where $\overset{\prime}{n}$, $\overset{\prime\prime}{n}$, $\overset{\prime\prime\prime}{n}$, &c. ſtand for $n-1$, $n-2$, $n-3$, &c. reſpectively.

PROP.

PROPOSITION II.

Suppoſing $\dot{n}$, $\underset{\prime\prime}{n}$, $\underset{\prime}{n}$, n, $\overset{\prime}{n}$, &c. *to be as in the laſt* Propoſition:

I ſay, if each Factor in the Denominator of the Fraction $\frac{1}{n\times\underset{\prime}{n}\times\underset{\prime\prime}{n}\times\underset{\prime\prime\prime}{n}\ldots\ldots\underset{\overset{\frown}{r-1}}{n}}$ *be diminiſhed by the common Difference, the Fraction it ſelf will be increaſed by* $\frac{r\dot{n}}{\overset{\prime}{n}n\underset{\prime}{n}\underset{\prime\prime}{n}\ldots\ldots\underset{\overset{\frown}{r-1}}{n}}$.

FOR, ſince n, diminiſhed by the common Difference, becomes equal to $\overset{\prime}{n}$; and $\underset{\prime}{n}$, diminiſhed by the ſame Difference, equal to n, *&c.* the new Value of the ſaid Fraction, ariſing from ſuch a Diminution of its Factors, will, it is evident, be equivalent to $\frac{1}{\overset{\prime}{n}n\underset{\prime}{n}\underset{\prime\prime}{n}\ldots\ldots\underset{\overset{\frown}{r-2}}{n}}$, under the firſt or given Values of thoſe Quantities; and therefore the Increaſe thereof muſt be $\frac{1}{\overset{\prime}{n}n\underset{\prime}{n}\ldots\ldots\underset{\overset{\frown}{r-2}}{n}} - \frac{1}{n\underset{\prime}{n}\underset{\prime\prime}{n}\ldots\ldots\underset{\overset{\frown}{r-1}}{n}}$ equal to

$$\frac{1}{n\underset{\prime}{n}\underset{\prime\prime}{n}\ldots\ldots\underset{\overset{\frown}{r-2}}{n}}\times\overline{\frac{1}{\overset{\prime}{n}}-\frac{1}{\underset{\overset{\frown}{r-1}}{n}}}=\frac{1}{n\underset{\prime}{n}\underset{\prime\prime}{n}\ldots\ldots\underset{\overset{\frown}{r-2}}{n}}\times\frac{\underset{\overset{\frown}{r-1}}{n}-\overset{\prime}{n}}{\overset{\prime}{n}\times\underset{\overset{\frown}{r-1}}{n}}$$ equal to

$\underset{\overset{\frown}{r-1}}{n}$

$$\frac{\underset{\underset{r-1}{\frown}}{n} - \dot{\overset{\prime}{n}}}{\overset{\prime}{n}\,n\,\underset{\prime}{n}\,\underset{\prime\prime}{n}\ldots\ldots\underset{\underset{r-1}{\frown}}{n}} = \frac{r\dot{n}}{\overset{\prime}{n}\,n\,\underset{\prime}{n}\,\underset{\prime\prime}{n}\ldots\ldots\underset{\underset{r-1}{\frown}}{n}}$$, becauſe $\underset{\underset{r-1}{\frown}}{n} - \dot{\overset{\prime}{n}}$ equal to $r\dot{n}$.

Q. E. D.

COROLLARY.

WHEREFORE, the Quantity, or Fraction, whoſe Increment is $\frac{r\dot{n}}{\overset{\prime}{n}\,n\,\underset{\prime}{n}\,\underset{\prime\prime}{n}\ldots\ldots\underset{\underset{r-1}{\frown}}{n}}$ being $\frac{1}{n\,\underset{\prime}{n}\,\underset{\prime\prime}{n}\,\underset{\prime\prime\prime}{n}\ldots\ldots\underset{\underset{r-1}{\frown}}{n}}$, the Quantity whoſe Increment is $\frac{1}{\overset{\prime}{n}\,n\,\underset{\prime}{n}\ldots\ldots\underset{\underset{r-1}{\frown}}{n}}$ muſt conſequently be $\frac{1}{r\dot{n}\times n\,\underset{\prime}{n}\,\underset{\prime\prime}{n}\ldots\underset{\underset{r-1}{\frown}}{n}}$: Whence to find the Value of a Fraction of this kind, from its Increment given, there ariſes this *Rule*. Strike out the leaſt Factor in the Denominator of the given Increment, and inſtead thereof put the Rectangle of the common Difference of the Factors into the Number of remaining Factors; the Reſult will be the Value that was to be found.

EXAMPLE.

LET it be required to find the Sum of the infinite Series $\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}$, &c. or one ſingle Fraction (if poſſible) that ſhall expreſs the Value of, &c. $\frac{1}{4.5} + \frac{1}{3.4} + \frac{1}{2.3}$. Here, if the required Fraction be conſidered as made up, or generated by a continual and regular Addition of the Terms, &c. $\frac{1}{4.5}$, $\frac{1}{3.4}$, $\frac{1}{2.3}$ of the propoſed Series, then $\frac{1}{1.2}$ being the next ſucceeding Term of the Progreſſion, or the Increment of the ſaid Fraction, the Fraction it ſelf, by the foregoing Rule, will be $= \frac{1}{1\times\overline{2-1}\times 2} = \frac{1}{2}$ In like manner will be found

$$\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \frac{1}{4.5}, \text{ \&c.} = 1.$$

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5}, \text{ \&c.} = \frac{1}{2.2}.$$

$$\frac{1}{1.2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{3.4.5.6}, \text{ \&c.} = \frac{1}{2.3.3}.$$

$$\frac{1}{1.2.3.4.5} + \frac{1}{2.3.4.5.6} + \frac{1}{3.4.5.6.7}. \text{ \&c.} = \frac{1}{2.3.4.4}.$$

$$\frac{1}{1.2.3.4.5.6} + \frac{1}{2.3.4.5.6.7} + \frac{1}{3.4.5.6.7.8}, \text{ \&c.} = \frac{1}{2.3.4.5.5}.$$

N. B. That in finding the Value of any Quantity by the Methods foregoing, it ought to be well conſidered, from the Nature of the Queſtion, whether that Quantity conſiſts barely of ſuch Fraction, Product, or Products, as are ſpecified in the Propoſitions, or thoſe joined to ſome invariable Quantity (as is done in Fluxions) and Allowances is to be made accordingly.

An Inveſtigation of Sir Iſaac Newton's *Theorem for finding the Sum of a Series of Numbers by means of their Differences.*

LET $a, b, c, d, e, f, g, h,$ *&c.* be the given Series of Numbers: Then, by taking each of them from the next ſucceeding, there will be $-a+b$, $-b+c$, $-c+d$, $-d+e$, $-f+g$, for the firſt Differences: Again, taking each of theſe Differences from its ſucceeding one, we have $a-2b+c$, $b-2c+d$, $c-2d+e$, $d-2e+f$, *&c.* for the ſecond Differences. In like manner the third Differences will be found $-a+3b-3c+d$, $-b+3c-3d+e$, $-c+3d-3e+f$, *&c.* and the fourth, $a-4b+6c-4d+e$, $b-4c+6d-4e+f$, *&c. &c.* Let the firſt Difference of the firſt Order be called $\overset{\prime}{D}$, the firſt of the ſecond Order $\overset{\prime\prime}{D}$, the firſt of the third Order $\overset{\prime\prime\prime}{D}$, *&c.* then we ſhall have $a=a$, $b=a+\overset{\prime}{D}$, $c=-a+2b+\overset{\prime\prime}{D}$, $d=a-3b+3c+\overset{\prime\prime\prime}{D}$, *&c.* and from thence by Subſtitution,

$$a=a$$

$$b=a+\overset{\prime}{D}$$

$$c=a+2\overset{\prime}{D}+\overset{\prime\prime}{D}$$

$$d=a+3\overset{\prime}{D}+3\overset{\prime\prime}{D}+\overset{\prime\prime\prime}{D}$$

$$e=a+4\overset{\prime}{D}+6\overset{\prime\prime}{D}+4\overset{\prime\prime\prime}{D}+\overset{\prime\prime\prime\prime}{D}.$$

where the Law of Continuation is manifeſt, the Unciæ of the Values of $c, d, e,$ *&c.* being thoſe of a Binomial, raiſed to the ſecond, third, fourth Powers, *&c.* Therefore, if n be put for the Number of Terms in the propoſed Series $a+b+c+d$, *&c.* whoſe Sum we are about to find, the Value of the next Term in the Progreſſion after the laſt in that Series, or that whoſe Place is defined by $n+1$, will, it is plain, be equal to $a+n\overset{\prime}{D}+n\times\frac{n-1}{2}\overset{\prime\prime}{D}+n\times\frac{n-1}{2}\times\frac{n-2}{3}\overset{\prime\prime\prime}{D}+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}\overset{\prime\prime\prime}{D}$, &c. And ſo much will the ſaid Sum be increaſed by augmenting n, the given Number of Terms by Unity; which Sum, therefore, by the firſt of the two foregoing *Propoſitions*, is $na+n\times\frac{n-1}{2}\overset{\prime}{D}+n\times\frac{n-1}{2}\times\frac{n-2}{3}\overset{\prime\prime}{D}+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}\overset{\prime\prime\prime\prime}{D}$, &c. *Q. E. I.*

EXAMPLE.

SUPPOSE *a*, *b*, *c*, *d*, &c. to be a Series of Squares, as 9, 16, 25, 36, &c. whoſe Roots are in Arithmetical Progreſſion; then will the firſt Differences be 7, 9, 11, 13, &c. the ſecond, 2, 2, 2, &c. the third, 0, 0, &c. &c. Therefore $a=9$, $\overset{\prime}{D}=7$, $\overset{\prime\prime}{D}=2$, $\overset{\prime\prime\prime}{D}=0$, $\overset{\prime\prime\prime\prime}{D}=0$, &c. and conſequently $na+n\times\frac{n-1}{2}\overset{\prime}{D}$, &c. equal $9n+n\times\frac{n-1}{2}\times7+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times2=9+16+25+36+49$, &c. continued to n Terms.

In the ſame manner the Sum of a Series of *Cubes*, *Biquadrates*, &c. may be found.

An eaſy and general Manner of inveſtigating the Sum of any recurring Series.

PROPOSITION.

Suppoſing p, q, r, s, &c. to be any Quantities, either poſitive or negative, and $A+B+C+D+E$, &c. a recurring Series, or one whoſe Terms A, B, C, &c. are ſo related, that any one of them, being multiplied by p, the next following, by q, the next in Order, by r, &c. the Sum of all the Products, thus ariſing, ſhall be equal to 0: To find (x) the Sum of ſuch a Series.

BECAUSE, by Suppoſition, $pA+qB+rC$, *&c.* is equal 0, $pB+qC+rD$, *&c.* equal 0, $pC+qD+rE$, *&c.* equal 0, *&c. &c.* it is evident, that the Sum of

$$\text{all these, or,}\left\{\begin{array}{l} pA+qB+rC+sD,\ \&c. \\ pB+qC+rD+sE,\ \&c. \\ pC+qD+rE+sF,\ \&c. \\ pD+qE+rF+sG,\ \&c. \\ pE+qF+rG+sH,\ \&c. \\ \&c.\quad \&c.\quad \&c.\quad \&c.\quad \&c. \end{array}\right\}\text{muſt conſequently be equal } 0.$$

Where, becauſe $A+B+C+D$, *&c.* is $=x$, $B+C+D$, *&c.* $=x-A$, $C+D+E$, *&c.* $=x-A-B$, *&c. &c.* the Value of the firſt Column towards the Left-hand will be px; that of the ſecond, $q\times\overline{x-A}$; that of the third, $r\times\overline{x-A-B}$; that of the fourth, $s\times\overline{x-A-B-C}$, *&c. &c*, and therefore

$px + q \times \overline{x - A} + r \times \overline{x - A - B} + s \times \overline{x - A - B - C}$, &c. or $px + qx - qA + rx - r \times \overline{A + B} + sx - s \times \overline{A + B + C}$, &c. $= 0$; hence $x = \frac{qA + r \times \overline{A+B} + s \times \overline{A+B+C} + t \times \overline{A+B+C+D}}{p+q+r+s+t+v,\ \&c.}$, &c. *Q. E. I.*

EXAMPLE I.

LET the Series $ay + y^2 + \frac{y^3}{a}$, &c. $(= x)$ be propoſed, where p equal $-y$, $q = a$, $r = 0$, $s = 0$, &c. $A = ay$, $B = yy$, &c, then will $x = \frac{qA + r \times \overline{A+B},\ \&c.}{p+q,\ \&c.}$, become $\frac{a^2 y}{a - y}$ in this Caſe.

EXAMPLE II.

LET $1 - \frac{3}{z^3} + \frac{5}{z^6} - \frac{7}{z^9} + \frac{9}{z^{12}}$, &c. $= x$, where $p = 1$, $q = 2z^3$, $r = z^6$, $s = 0$, $t = 0$, &c. Then A being $= 1$, $B = -\frac{3}{z^3}$, &c. $\frac{qA + r \times \overline{A+B},\ \&c.}{p+q,\ \&c.}$ $(= x)$ will here become $\frac{z^6 - z^3}{z^6 + 2z^3 + 1}$.

A Method for finding the Sum of a Series of Powers, &c.

PROPOSITION

To find the Sum of any Series of Powers whoſe Roots are in Arithmetical Progreſſion, as $\overline{m+d}|^{n}+\overline{m+2d}|^{n}+\overline{m+3d}|^{n}\ldots\ldots\ldots x^{n}$, *m, d, and n, being any Numbers whatſoever.*

LET $Ax^{n+1}+Bx^{n}+Cx^{n-1}+Dx^{n-2}+Ex^{n-3}+Fx^{n-4}$, &c. — K, if poſſible, be always equal to $\overline{m+d}|^{n}+\overline{m+2d}|^{n}\ldots\ldots x^{n}$, and A, B, C, &c. determinate Quantities: Then, if any other Number in the Progreſſion $m+d$, $m+2d$, $m+3d\ldots\ldots x+d$, $x+2d$, $x+3d$, &c. as $x+d$, be ſubſtituted inſtead of x, the Equality will ſtill continue; and we ſhall have $A\times\overline{x+d}|^{n+1}+B\times\overline{x+d}|^{n}+C\times\overline{x+d}|^{n-1}+D\times\overline{x+d}|^{n-2}$ &c. — K equal $\overline{m+d}|^{n}+\overline{m+2d}|^{n}\ldots\ldots\ldots\overline{x+d}|^{n}$; from which taking the former Equation there remains $A\times\overline{\overline{x+d}|^{n+1}-x^{n+1}}+B\times\overline{\overline{x+d}|^{n}-x^{n}}+C\times\overline{\overline{x+d}|^{n-1}-x^{n-1}}$, &c. $=\overline{x+d}|^{n}$, ſhewing how much each Side is increaſed by augmenting the Number of Terms in the given Series by Unity; where, by tranſpoſing $\overline{x+d}|^{n}$, and throwing the ſeveral Powers of $x+d$ into Series, we have

— x

$$\left.\begin{array}{llll}
-x^n & -nx^{n-1}d & -n\times\frac{n-1}{2}\times x^{n-2}d^2 & -n\times\frac{n-1}{2}\times\frac{n-2}{3}x^{n-3}d^3,\ \&c. \\
\overline{n+1}\,Ax^nd+ & n+1\times\frac{n}{2}Ax^{n-1}d^2+ & n+1\times\frac{n}{2}\times\frac{n-1}{3}Ax^{n-2}d^3+ & n+1\times\frac{n}{2}\times\frac{n-1}{3}\times\frac{n-2}{4}Ax^{n-3}d^4,\ \&c. \\
\bullet & +\ nBx^{n-1}d^2+ & n\times\frac{n-1}{2}Bx^{n-2}d^3+ & n\times\frac{n-1}{2}\times\frac{n-2}{3}Bx^{n-3}d^4,\ \&c. \\
\bullet & \bullet & +\ n-1\times Cx^{n-2}d^3+ & n-1\times\frac{n-2}{2}Cx^{u-3}d^4,\ \&c. \\
\bullet & \bullet & \bullet & n-2\,Dx^{n-3}d^4,\ \&c. \\
 & & & \&c.
\end{array}\right\}=0.$$

From which, by equating the homologous Terms, A will come out $=\frac{1}{\overline{n+1}\times d}$, $B=\frac{1}{2}$, $C=\frac{nd}{3.4}$, $D=0$, $E=-\frac{n\times\overline{n-1}\times\overline{n-2}\times d^{3}}{2.3.4.5.6}$, $F=0$, $G=\frac{n\times\overline{n-1}\times\overline{n-2}\times\overline{n-3}\times\overline{n-4}\times d^{5}}{2.3.4.5.6.7.6}$ $H=0$, *&c.* wherefore the Values of A, B, C, *&c.* being ſo aſſigned, the whole Expreſſion, or its Equal $-\overline{x+d}|^{n}$ $+A\times\overline{\overline{x+d}|^{n+1}-x^{n+1}}+B\times\overline{\overline{x+d}|^{n}-x^{n}}$, *&c.* muſt be equal 0, and conſequently $A\times\overline{\overline{x+d}|^{n+1}-x^{n+1}}+\overline{B\times\overline{x+d}|^{n}-x^{n}}$, *&c.* $=\overline{x+d}$; that is, let x and n be what they will, the foreſaid Increments of $Ax^{n-1}+Bx^{n}+Cx^{n-1}$, *&c.* $-K$ and $\overline{m+d}|^{n}+\overline{m+2d}|^{n}$, *&c.* will, under the above aſſigned Values of A, B, *&c.* be equal to one another: Therefore, if K be taken equal $Am^{n+1}+Bm^{n}+Cm^{n-1}$, *&c.* ſo that when x equal m, or the propoſed Series is equal to Nothing, $Ax^{n+1}+Bx^{n}$, *&c.* $-K$ may be alſo $=0$, it is manifeſt, that theſe two Expreſſions, as they are increaſed alike, will, in all other Circumſtances, be equal; that is, let x be what it will $Ax^{n+1}+Bx^{n}+Cx^{n-1}+Dx^{n-2}$, *&c.* $-Am^{n+1}-Bm^{n}-Cm^{n-1}-Dm^{n-2}$, *&c.* under the ſaid Values of A, B, C, *&c.* will be always equal to $\overline{m+d}|^{n}+\overline{m+2d}|^{n}+\overline{m+3d}|^{n}\ldots x^{n}$; which Values being therefore ſubſtituted, there will be $\frac{x^{n+1}}{\overline{n+1}\times d}+\frac{x^{n}}{2}+\frac{dnx^{n-1}}{3.4}-\frac{n\times\overline{n-1}\times\overline{n-2}d^{3}x^{n-3}}{2.3.4.5.6}$

+

$$+\frac{n\times n-1\times n-2\times n-3\times n-4}{2.3.4.5.6.7.6}d^{5}x^{n-5}-\frac{n\times n-1\times n-2\times n-3,\ \&c.}{2.3.4.5.6.7.8.5.6.}d^{7}x^{n-7}$$

$$+\frac{n\times n-1\times n-2\times n-3\times n-4\times n-5\times n-6\times n-7\times n-8}{2.3.4.5.6.7.8.9.11.12.}d^{9}x^{n-9},\ \&c.\ -$$

$$\frac{m^{n+1}}{\overline{n+1}\times d}-\frac{m^{n}}{2}-\frac{ndm^{n-1}}{3.4}+\frac{n\times n-1\vee n-2}{2.3.4.5.6}\times d^{3}m^{n-3},\ \&c.$$

$$=\overline{m+d}|^{n}+\overline{m+2d}|^{n}+\overline{m+3d}|^{n}\ \ldots\ldots\ x^{n}$$

Q. E. I.

COROL. I.

HENCE, if n be a whole poſitive Number, and m be taken equal 0; then all the Terms in the ſecond Series $-\frac{m^{n+1}}{\overline{n+1}\times d}-\frac{m^{n}}{2}+\frac{ndm^{n-1}}{3.4}$, &c. vaniſhing when n is even, and all but that where the Exponent of m is nothing, when odd, we ſhall, in this Caſe, have $d^{n}+\overline{2d}|^{n}+\overline{3d}|^{n}+\overline{4d}|^{n}\ \ldots\ldots\ x^{n}$ barely equal to $\frac{x^{n+1}}{\overline{n+1}\times d}+\frac{x^{n}}{2}+\frac{ndx^{n-1}}{3.4}-\frac{n\times n-1\times n-2d^{3}x^{n-3}}{2.3.4.5.6}$, &c. the firſt Series continued 'till it terminates, provided that the laſt Term, when n is an odd Number, be rejected.

COROL. II.

WHEREFORE, by taking d equal to 1, and n equal to 2, 3, 4, 5, &c. ſucceſſively, we have

$$1 + 2 + 3 + 4 + 5 \ldots + x = \frac{x^2}{2} + \frac{x}{2}$$

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 \ldots + x^2 = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6}$$

$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 \ldots + x^3 = \frac{x^4}{4} + \frac{x^3}{2} + \frac{x^2}{4}$$

$$1^4 + 2^4 + 3^4 + 4^4 + 5^4 \ldots + x^4 = \frac{x^5}{5} + \frac{x^4}{2} + \frac{x^3}{3} - \frac{x}{30}$$

$$1^5 + 2^5 + 3^5 + 4^5 + 5^5 \ldots + x^5 = \frac{x^6}{6} + \frac{x^5}{2} + \frac{5x^4}{12} - \frac{x^2}{12}$$

$$1^6 + 2^6 + 3^6 + 4 + 5^6 \ldots + x^6 = \frac{x^7}{7} + \frac{x^6}{2} + \frac{x^5}{2} - \frac{x^3}{6} + \frac{x}{42}$$

&c. &c.

COROL. III.

MOREOVER, if d be taken equal to 1, and m equal to 1, our general Equation will become $2^n + 3^n + 4^n \ldots + x^n = \frac{x^{n+1}}{n+1} + \frac{x^n}{2} + \frac{n x^{n-1}}{3 \cdot 4}$; &c. $- \frac{1}{n+1} - \frac{1}{2} - \frac{n}{3 \cdot 4} + \frac{n \times \overline{n-1} \times \overline{n-2}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$, &c. each Side of which being increaſed by Unity, and the whole multiplied by d^n gives $d^n + \overline{2d}|^n + \overline{3d}|^n + \overline{4d}|^n \ldots + \overline{dx}|^n = d^n$ into $\frac{x^{n+1}}{n+1} + \frac{x^n}{2} + \frac{n x^{n-1}}{3 \cdot 4} - \frac{n \times \overline{n-1} \times \overline{n-2}\, x^{n-3}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$, &c. $- \frac{1}{n+1} + \frac{1}{2} - \frac{n}{3 \cdot 4} + \frac{n \times \overline{n-1} \times \overline{n-2}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$, &c.

EXAMPLE I.

LET it be required to find the Sum of a Series, conſiſting of 100 Cube Numbers, whoſe Roots are, $\frac{1}{2}$, 1, $\frac{3}{2}$, 2, $\frac{5}{2}$, 3, &c.

Here *d*, the common Difference of Roots, being equal $\frac{1}{2}$, $n = 3$, and $x = 0$, let theſe Values be ſubſtituted in the Equation in *Cor.* II. and it will become $\left(\overline{\frac{1}{2}}\Big|^{3}\right.$ *in*, $\frac{\overline{100}|^{4}}{4} + \frac{\overline{100}|^{3}}{2} + \frac{\overline{100}|^{2}}{4} = \Big)$ 3187812.5, the Number that was to be found.

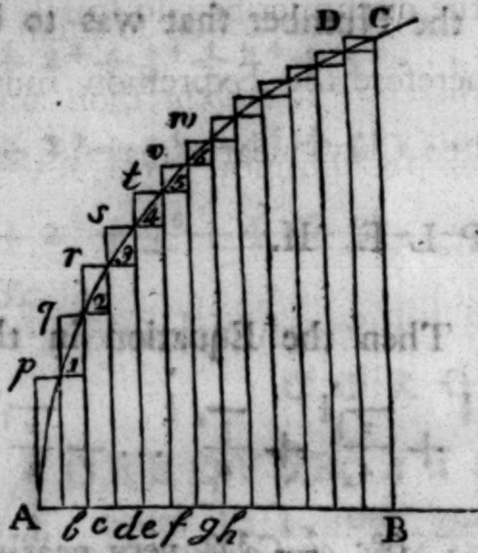
EXAMPLE II.

LET $n = \frac{1}{2}$, $= \frac{1}{4}$. Then the Equation in the laſt *Corollary* will become $\overline{\frac{1}{4}}\Big|^{\frac{1}{2}} + \overline{\frac{2}{4}}\Big|^{\frac{1}{2}} + \overline{\frac{3}{4}}\Big|^{\frac{1}{2}} \ldots + \overline{\frac{x}{4}}\Big|^{\frac{1}{2}}$ $= \overline{\frac{1}{4}}\Big|^{\frac{1}{2}} \times \overline{\frac{2 \times x^{\frac{3}{2}}}{3} + \frac{x^{\frac{1}{2}}}{2} + \frac{1}{24 \times x^{\frac{1}{2}}}, \&c.} - \frac{339}{1920}$ very nearly; ſo that, taking x equal 4, it will be $\overline{\frac{1}{4}}\Big|^{\frac{1}{2}} + \overline{\frac{2}{4}}\Big|^{\frac{1}{2}} + \overline{\frac{3}{4}}\Big|^{\frac{1}{2}} + 1 = 3.0731$; which differs from the true Value by leſs than $\frac{1}{10000}$; and if more Terms had been uſed, the Anſwer would ſtill have been more exact; but never can come *accurately* true, when n is negative or a Fraction, becauſe then both Series run on *ad infinitum*.

SCHOLIUM.

THE Theorems, above found, are not only uſeful in finding the Sum of a Series of Powers, but may be of ſervice alſo in the Quadrature of Curves, &c. eſpecially as the Concluſions will be accurately true, and the Reaſoning thereupon ſcientific.

This I ſhall endeavour to ſhew by the following Inſtance; wherein A C, being ſuppoſed a Curve, whoſe Equation is $y = z^n$ (A B being equal z, and C B equal y) the Area A B C is required.



Let A B be divided into any Number, x, of equal Parts, as Ab, bc, cd, *&c.* and from the Points of Diviſion let Perpendiculars be raiſed, cutting the Curve in the Points, 1, 2, 3, *&c.* and having made p1, q2, r3, s4, *&c.* parallel to A B, let the Baſe Ab, bc, cd, *&c.* of each of the Rectangles pb, qc, rd, *&c.* be repreſented by d: Then b1, c2, d3, *&c.* the Heights of thoſe Rectangles, being Ordinates to the Curve, will be d^n, $\overline{2d}|^n$, $\overline{3d}|^n$, *&c.* reſpectively, each of which $\because$ being multiplied by d, the common Baſe, and the Sum of all the Products taken, will give d into $d^n + \overline{2d}|^n + \overline{3d}|^n \ldots \overline{xd}|^n$, (= A$p1q2r$, *&c.* CBA) for the Area of the whole circumſcribing Polygon; and this Series, according to the abovesaid Theorem (*Cor.* III.) is equal to d^{n+1} *in*, $\frac{nx^{n+1}}{n+1} + \frac{x^n}{2}$ *&c.* $= \frac{\overline{dx}|^{n+1}}{n+1} + \frac{d \times \overline{dx}|^n}{2}$, *&c.* or, becauſe $dx = z$, it will be $= \frac{z^{n+1}}{n+1} + \frac{dx^n}{2}$, *&c.* Now, if from this the Difference of the Inſcribed and circumſcribed Polygons, or the

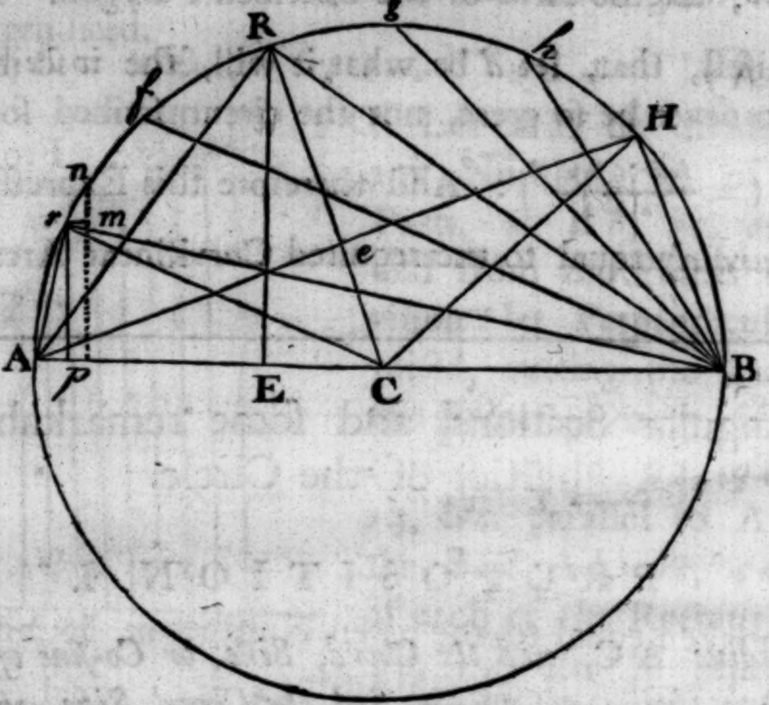
Rectangle BD $= dz^n$ be taken, there will remain $\frac{z^{n+1}}{n+1} - \frac{dzn}{2}$, for the Area of the inſcribed Polygon. Hence, it is manifeſt, that, let d be what it will, the inſcribed Polygon can never be ſo great, nor the circumſcribed ſo ſmall, as $\frac{z^{n+1}}{n+1}$ $\left(= \frac{AB \times BC}{n+1}\right)$: And therefore this Expreſſion muſt be *accurately* equal to the required Curvilinear Area ACB.

Of Angular Sections, and ſome remarkable Properties of the Circle.

PROPOSITION I.

The Radius AC, *and the Chord, Sine, or Co-ſine of an Arc, as* Ar, *being given; to find the Chord, Sine, or Co-ſine of* AR $= m \times$ Ar, *a Multiple of that Arc.*

LET RH be taken = AR, and the whole Arc AH be divided into as many equal Parts, Ar, rf, &c. as there be Units in $2m$; and the Chords Br, Bf, &c. are drawn, as alſo the Radii Cr, CR, CH, and the Perpendiculars rp, RE; calling AC, 1; Br, y; Cp, x; CE, X; rp, u; RE, U; Ar^2, z; and AH^2 = Z: Then, becauſe any one of thoſe Chords, as Bf, is to Br + BR, the Sum of the 2 next it, as BC to Br, by a known Property of the Circle, we ſhall have $y \times \mathrm{B}f = \mathrm{B}r + \mathrm{BR}$, or $y \times \mathrm{B}f - \mathrm{B}r = \mathrm{BR}$; and for the very ſame Reaſon, $y \times \mathrm{BR} - \mathrm{B}f = \mathrm{B}g$, and $y \times \mathrm{B}g - \mathrm{BR} = \mathrm{B}b$, &c. &c. Hence, it appears, that the Values of the Chords Bf, BR, &c. (which to a Radius equal A B, will be Co-ſines of the Angles A B f,



A B R, &c.) may be readily had one after another, by taking continually the Product of the laſt by y, *minus* the laſt but one, for the next following: And thus are had,

$$y^2 - 2 = Bf,$$
$$y^3 - 3y = BR,$$
$$y^4 - 4y^2 + 2 = Bg,$$
$$y^5 - 5y^3 + 5y = Bh,$$
$$y^6 - 6y^4 + 9y^2 - 2 = BH,$$
&c. &c.

And generally, ſuppoſing $\acute{A}y^{n-1} - \acute{B}y^{n-3} + \acute{C}y^{n-5}$, &c. to denote any one Chord of the foregoing Order, and A y

—

$-By^{n-2}+Cy^{n-4}$, &c. the next to it; then the Chord next following theſe will be $Ay^{n+1}-By^{n-1}+Cy^{n-3}$ &c. $-\acute{A}y^{n-1}+\acute{B}y^{n-3}-\acute{C}y^{n-5}$, &c. $=Ay^{n+1}$ $\left.\begin{matrix}B\\ \acute{A}\end{matrix}\right\}y^{n-1}+\left.\begin{matrix}C\\ \acute{B}\end{matrix}\right\}y^{n-3}+\left.\begin{matrix}D\\ \acute{C}\end{matrix}\right\}y^{n-5}$, &c. From which (by the Method of Increments foregoing) A will come out $=1$, $B=n$, $C=n\times\frac{n-3}{2}$, $D=n\times\frac{n-4}{2}\times\frac{n-5}{3}$, $E=n\times\frac{n-5}{3}\times\frac{n-6}{3}\times\frac{n-7}{4}$, &c. and conſequently $Ay^{n}-By^{\frac{n-2}{2}}+Cy^{n-4}$, &c. $=y^{n}-ny^{n-2}+n\times\frac{n-3}{2}y^{n-4}-n\times^{n-4}\times\frac{n-5}{2}y^{n-6}+n\times\frac{n-5}{2}\times\frac{n-6}{3}\times\frac{n-7}{4}y^{n-8}$, &c. wherein if n be taken equal to the given Number m, it will become $y^{m}-my^{m-2}+m\times\frac{m-3}{2}y^{m-4}$, &c. equal BR; but if n be equal $2m$, then it will be $y^{n}-ny^{n-2}+n\times\frac{n-3}{2}y^{n-4}$, &c. equal BH; where the Series are to be continued till the Exponents become negative. Hence, becauſe Bf is equal $2x$, and the Arc $AH=m\times Af$, it follows, that the Chord HB will be $=\overline{2x}|^{m}-m\times\overline{2x}|^{m-2}+m\times\frac{m-3}{2}\times\overline{2x}|^{m-4}$, &c. and therefore, X ($=$ CE) the required Co-ſine being equal $\frac{1}{2}$ HB, we have $X=\frac{\overline{2x}|^{m}}{2}-\frac{m}{2}\times\overline{2x}|^{m-2}+\frac{m}{2}\times\frac{m-3}{2}\times\overline{2x}|^{m-4}-\frac{m}{2}\times\frac{m-4}{2}\times\frac{m-5}{3}\times\overline{2x}|^{m-6}+\frac{m}{2}\times\frac{m-5}{2}\times\frac{m-6}{3}\times\frac{m-7}{4}\times\overline{2x}|^{m-8}$ &c. ſhewing the Relation of the Co-ſines; from whence

U $(=\sqrt{1-X^2})$ comes out $=\sqrt{1-xx}$, *in*, $\overline{2x}|^{m-1} - \frac{m-2}{1} \times \overline{2x}|^{m-3} + \frac{m-3}{1} \times \frac{m-4}{2} \times \overline{2x}|^{m-5} - \frac{m-4}{1} \times \frac{m-5}{2} \times \frac{m-6}{3} \times \overline{2x}|^{m-7}$, &c. Furthermore, becauſe $\frac{BH}{2}$ is equal CE, $= \frac{y^n - ny^{n-2} + n \times \frac{n-3}{2} y^{n-4}, \text{ \&c.}}{2}$, A E will be equal to $\frac{-y^n + ny^{n-2} - n \times \frac{n-3}{2} y^{n-4}, \text{ \&c.}}{2} + 1$; and therefore AE × AB equal to $-y^n + ny^{n-2} - n \times \frac{n-3}{2} y^{n-4}$ &c. $+2 = Z$, where, if inſtead of yy, its Equal $-z+4$ $(AB^2 - Ar^2)$ be ſubſtituted, it will become $Z = \pm z^{\frac{n}{2}} \mp nz^{\frac{n-2}{2}} \pm n \times \frac{n-3}{2} z^{\frac{n-4}{2}}$, &c. equal $\pm z^m \mp 2mz^{m-1} \pm 2m \times \frac{2m-3}{2} \times z^{m-2} \mp 2m \times \frac{2m-4}{3} \times \frac{2m-5}{3} \times z^{m-3} \pm 2m \times \frac{2m-5}{2} \times \frac{2m-6}{3} \times \frac{2m-7}{4} \times z^{m-4}$, &c. continued to as many Terms as there are Units in *m*. *Q. E. I.*

Otherwiſe,

Let the Lines rp, RE, be conſidered in a flowing State, and (mn) as equal to $\dot{x}$; then we ſhall have $\sqrt{1-xx}$ $(pr) : 1\ (Cr) :: \dot{x} : \frac{\dot{x}}{\sqrt{1-x|x}}$ equal rn; and this being the Fluxion of the Arc Ar, that of AR (equal $m \times Ar$) will be $\frac{m\dot{x}}{\sqrt{1-xx}}$; which, for the very ſame Reaſon that $\frac{\dot{x}}{\sqrt{1-xx}}$ is the Fluxion of the A r, muſt be equal to $\frac{\dot{X}}{\sqrt{1-X^2}}$: Whence, equally multiplying the two Denominators by $\sqrt{-1}$, we get $\frac{m\dot{x}}{\sqrt{xx-1}} = \frac{\dot{X}}{\sqrt{X^2-1}}$; where, taking the Fluent on each Side, there comes out, either, Log. $X+\sqrt{X^2-1} = m \times$ Log. $x + \sqrt{xx-1}$, or, Log. $X-\sqrt{XX-1} = m$ Log. $x - \sqrt{xx-1}$; wherefore, $X+\sqrt{X^2-1}$ and $\overline{x+\sqrt{xx-1}}\big|^{m}$, as alſo, $X - \sqrt{X^2-1}$ and $\overline{x-\sqrt{xx-1}}\big|^{m}$, the Numbers correſponding to thoſe Logarithms muſt be equal: Hence, by adding together the two Equations, we have $2X = \overline{x+\sqrt{xx-1}}\big|^{m} + \overline{x+\sqrt{xx-1}}\big|^{m}$, and by taking their Difference, $2\sqrt{X^2-1} = \overline{x+\sqrt{xx-1}}\big|^{m} - \overline{x-\sqrt{xx-1}}\big|^{m}$; from whence, by expanding the latter Part of each of the Equations into Series, and dividing the whole by 2, there will come out $X = x^{m} + m \times \frac{m-1}{2} x^{m-2} \times \overline{xx-1} + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}\ x^{m-4} \times \overline{xx-1}\big|^{2}$, &c. and $\sqrt{X^2-1} = \sqrt{xx-1}$ *in*, $mx^{m-1} + m \times \frac{m-1}{2} \times \frac{m-2}{3} x^{m-3} \times \overline{xx-1}$, &c. the former of which being reduced into ſimple Terms, gives $X = \frac{\overline{2x}\big|^{m}}{2} - \frac{m}{2} \times \overline{2x}\big|^{m-2} + \frac{m}{2} \times \frac{m-3}{2} \times \overline{2x}\big|^{m-4}$, &c. the very ſame as above found. And the latter, by multiplying by $\sqrt{-1}$, to

take

take away the imaginary Quantities, and ſubſtituting U and u inſtead of their Equals $\sqrt{1-X^2}$, $\sqrt{1-xx}$, becomes $U = u$ *in*, $m \times \overline{1-uu}|^{\frac{m-1}{2}} + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times -u^2 \times \overline{1-uu}|^{\frac{m-3}{2}} + m \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \overline{1-uu}|^{\frac{m-5}{2}} \times u^4$, *&c.* which, in like manner, being reduced into ſimple Terms, will be $U = mu - m \times \frac{m^2-1}{2.3} \times u^3 + m \times \frac{m^2-1}{2.3} \times \frac{m^2-9}{4.5} \times u^5 - m \times \frac{m^2-1}{2.3} \times \frac{m^2-9}{4.5} \times \frac{m^2-25}{6.7} \times u^7$, *&c.* *Q. E. I.*

COROL. I.

BECAUSE the laſt Equation, as appears from the Proceſs, will hold as well when m is a Fraction as when a whole Number; let m, or the Multiple Arch A R ($= m \times$ A r) be ſuppoſed indefinitely ſmall; then will $mu - m \times \frac{m^2-1}{2.3} \times u^3 + m \times \frac{m^2-1}{2.3} \times \frac{m^2-9}{4.5} \times u^5$, *&c.* the Sine of that Arch, or the Arch it ſelf (which in this Caſe may be conſidered as equal to it) become $mu + \frac{mu^3}{2.3} + \frac{9mu^5}{2.3.4.5} + \frac{9 \times 25 u^7}{2.3.4.5.6.7.}$, *&c.* and therefore the Arch A r $\left(= \frac{AR}{m}\right)$ whoſe Sine is u, will, it is manifeſt, be $= u + \frac{u^3}{2.3} + \frac{3.3u^5}{2.3.4.5} + \frac{3.3.5.5u^7}{2.3.4.5.6.7} + \frac{3.3.5.5.7.7u^9}{2.3.4.5.6.7.8.9}$, *&c.*

COROL.

COROL. II.

IF Ar be ſuppoſed indefinitely ſmall, and m indefinitely great, ſo that the Multiple Arch $m \times Ar$ $(=A)$ may be a given Quantity; then ſince u may be conſidered as equal to Ar, mu will be equal to A, and $mu - m \times \frac{m^2-1}{2.3} \times u^3$, &c. the Sine of A, equal to $mu - \frac{m^3u^3}{2.3} + \frac{m^5u^5}{2.3.4.5}$ or $A - \frac{A^3}{2.3} + \frac{A^5}{2.3.4.5} - \frac{A^7}{2.3.4.5.6.7}$, &c. becauſe 1, 9, 25, &c. in the Factors m^2-1, m^2-9, &c. may here be rejected as indefinitely ſmall in compariſon of m^2.

SCHOLIUM.

BECAUSE $\overline{x + \sqrt{xx-1}}\vert^m + \overline{x - \sqrt{xx-1}}\vert^m$ is found above to be univerſally $= \overline{2x}\vert^m - m \times \overline{2x}\vert^{m-2} + m \times \frac{m-3}{2} \times \overline{2x}\vert^{m-4} - m \times \frac{m-4}{2} \times \frac{m-5}{3} \times \overline{2x}\vert^{m-6}$, &c. it is evident, by Inſpection, that $\overline{x + \sqrt{xx+1}}\vert^m + \overline{x - \sqrt{xx+1}}\vert^m$ will be $= \overline{2x}\vert^m + m \times \overline{2x}\vert^{m-2} + m \times \frac{m-3}{2} \times \overline{2x}\vert^{m-4}$, &c. and ∴ $\overline{\frac{y}{2} + \sqrt{\frac{yy}{4} + rr}}\Big\vert^m + \overline{\frac{y}{2} - \sqrt{\frac{yy}{4} + rr}}\Big\vert^m = y^m + my^{m-2}r^2 + m \times \frac{m-3}{2} y^{m-4} r^4$, &c. (by ſubſtituting $\frac{y}{2}$ in the room of x, and rr in that of Unity) let r and y be what they will: Therefore, if $y^m + my^{m-2}r^2 +$

m

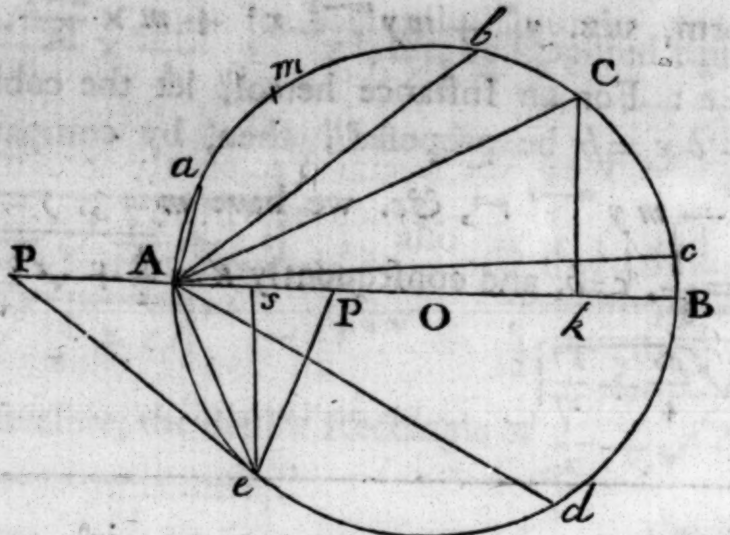
$m \times \frac{m-3}{2} y^{m-4} r^4 + m \times \frac{m-4}{2} \times \frac{m-5}{3} y^{m-6} r^6 + m \times \frac{m-5}{2} \times \frac{m-6}{3} \times \frac{m-7}{4} y^{m-8} r^8$, &c. be ſuppoſed equal to ſome given Quantity c, there will be given $\overline{\frac{y}{2} + \sqrt{\frac{yy}{4} + rr}}\Big|^{m} + \overline{\frac{y}{2} - \sqrt{\frac{yy}{4} + rr}}\Big|^{m}$, alſo $= c$; and therefore $\overline{\frac{y}{2} + \sqrt{\frac{yy}{4} + rr}}\Big|^{2m} - 2r^{2m} + \overline{\frac{y}{2} - \sqrt{\frac{yy}{4} + rr}}\Big|^{2m} = cc$; wherefore, the double Rectangle of $\overline{\frac{y}{2} + \sqrt{\frac{yy}{4} + rr}}\Big|^{m}$ into $\overline{\frac{y}{2} - \sqrt{\frac{yy}{4} + rr}}\Big|^{m}$ being $-2r^{2m}$, the Square of $\overline{\frac{y}{2} + \sqrt{\frac{yy}{4} + rr}}\Big|^{m} - \overline{\frac{y}{2} - \sqrt{\frac{yy}{4} + rr}}\Big|^{m}$ will be $= cc + 4r^{2m}$, and conſequently $\overline{\frac{y}{2} + \sqrt{\frac{yy}{4} + rr}}\Big|^{m} - \overline{\frac{y}{2} - \sqrt{\frac{yy}{4} + rr}}\Big|^{m} = \sqrt{cc + 4r^{2m}}$; which Equation added to the firſt gives, $2 \times \overline{\frac{y}{2} + \sqrt{\frac{yy}{4} + rr}}\Big|^{m} = c + \sqrt{cc + 4r^{2m}}$; and ſubtracted therefrom, $2 \times \overline{y - \sqrt{\frac{yy}{4} + rr}}\Big|^{m} = c - \sqrt{cc + 4r^{2m}}$; whence we have $\frac{y}{2} + \sqrt{\frac{yy}{4} + rr} = \overline{\frac{c}{2} + \sqrt{\frac{cc}{4} + r^{2m}}}\Big|^{\frac{1}{m}}$, and $\frac{y}{2} - \sqrt{\frac{cc}{4} + r^{2m}} = \overline{\frac{c}{2} - \sqrt{\frac{cc}{4} + r^{2m}}}\Big|^{\frac{1}{m}}$ and therefore $y = \overline{\frac{c}{2} + \sqrt{\frac{cc}{4} + r^{2m}}}\Big|^{\frac{1}{m}} + \overline{\frac{c}{2} - \sqrt{\frac{cc}{4} + r^{rm}}}\Big|^{\frac{1}{m}}$:

Which may be uſeful and ſerve as a Theorem for the Solution of certain Kind of adfected Equations, comprehended in this Form, *viz.* $y^{m} + m y^{m-2} r^{2} + m \times \frac{m-3}{2} y^{m-4} r^{4}$, &c. $= c$: For an Inſtance hereof, let the cubic Equation $x^{3} + b x = b$ be propoſed; then, by comparing this with $y^{m} - m y^{m-2} r^{2}$, &c. we have $m = 3$, $y = x$, $m r^{2} = b$, or $r r = \frac{b}{3}$, $c = b$, and conſequently $x = \overline{\frac{b}{2} + \sqrt{\frac{bb}{4} + \frac{b^{3}}{27}}}\Big|^{\frac{1}{3}} + \overline{\frac{b}{2} - \sqrt{\frac{bb}{4} + \frac{b^{3}}{27}}}\Big|^{\frac{1}{3}}$

PROPOSITION II.

If on the Diameter A B, *from any Point* C, *in the Circle* A C B, *whoſe Centre is* O, *the Perpendicular* C *k be let fall, and the Arc* A C *be divided into any Number, m, of equal Parts, as* A *a*, *a m*, &c. *and if the whole Periphery be alſo divided into the ſame Number of equal Parts, beginning at the Point a, as a b, b c, c d*, &c. *and from any Point* P, *in the Diameter* A B, *or* A B *produced, Lines be drawn to the Points a, b, c*, &c. *I ſay*, $Pa^{2} \times Pb^{2} \times Pc^{2} \times Pd^{2}$, &c. *the continual Product of the Squares of all thoſe Lines will be equal to* $AO^{2m} \mp AO^{m-1} \times 2Ok \times OP^{m} + PO^{2m}$.

PUT AO = to 1, PO = to x, AP^{2} = to $\overline{1 \backsim x}|^{2}\, v$, O$k$ = to b, $2m$ = to n, and the Square of any one of the Chords A*a*, A*b*, A*c*, A*d*, &c. equal to z: Then, ſince any one of the correſponding Arcs A*a*, A*b*, A*bc*, &c. reckoned forward a certain Number of Times, brings us to the ſame Point C, or, is equal to AC, or AC *plus* a certain Number



ber of Times the whole Periphery, it appears from the laſt *Propoſition* that $\pm z^{m} \mp n z^{m-1} \pm n \times \frac{n-3}{2} z^{m-2} \mp n \times \frac{n-4}{2} \times \frac{n-5}{3} z^{m-3}$, *&c.* continued to m Terms, is $= AC^2$, or becauſe AC^2 is $= 2 + 2b$ $(AB \times Ak)$ it will be $z^{m} - n z^{m-1} + n \times \frac{n-3}{2} z^{m-2} - n \times \frac{n-4}{2} \times \frac{n-5}{3} z^{m-3} \ldots\ldots \mp \overline{2 + 2b} = 0$, let z ſtand for the Square of which of thoſe Chords you will: Wherefore, the Roots of this Equation being the Squares of the Chords Aa, Ab, Ac, *&c.* they muſt be all poſitive, their Sum $= n$, the Sum of their Products $n \times \frac{n-3}{2}$, of their Solids $n \times \frac{n-4}{2} \times \frac{n-5}{3}$, *&c.* by common Algebra. Now, if se be made perpendicular to AB, we ſhall have $AP^2 + Ae^2 \pm AP \times 2As = Pe^2 = AP^2 + Ae^2 \pm AP \times \frac{Ae^2}{AO} = AP^2 + \frac{PO}{AO} \times \overline{Ae}|^2$, which in Species, is $Pe^2 = v+x\times\overline{Ae}|^2$: And, for the very ſame Reaſons, $Pb^2 = v+x\times Ab^2$, $Pc^2 = v+x\times Pc^2$, *&c.* therefore the continual Product of $v+x\times Aa^2$ into $v+x\times Ab^2$ into $v+x\times Ac^2$, *&c.* is equal to $Pa^2\times Pb^2\times Pc^2$, *&c.* But in the former of theſe Products, it is evident, that when the ſeveral Factors are actually drawn into one another, the Co-efficient of the firſt Term or higheſt Power of v, will be 1; of the next inferior Power, the Sum of all the aboveſaid Roots Aa^2, Ab^2, *&c.* into x, of the next following, the Sum of all their Products into x^2, *&c.* and, therefore, the Sum of thoſe Roots being already found $= n$, their Products $= n\times\frac{n-3}{2}$, *&c.* we have $v^m + nxv^{m-1} + n\times\frac{n-3}{2}x^2v^{m-2} + n\times\frac{n-4}{2}\times\frac{n-5}{3}x^3v^{m-3} + n\times\frac{n-5}{2}\times\frac{n-6}{3}\times\frac{n-7}{4}x^4v^{m-4} \ldots\ldots + \overline{2+2b}\times x^m = Pa^2\times Pb^2\times Pc^2$, *&c.* Or, by ſubſtituting for v, its Equal $\overline{1\backsim x}|^2$ it will be $\overline{1\backsim x}|^n + nx\times\overline{1\backsim x}|^{n-2} + n\times\frac{n-3}{2}x^2\times\overline{1\backsim x}|^{n-4}\ldots\ldots + \overline{2+2b}\times x^m = Pa^2\times Pb^2\times Pc^2$, *&c.* (becauſe $2m = n$): This in ſimple Terms is

$$\left.\begin{array}{l}
1 - nx + n\times\frac{n-1}{2}x^2 - n\times\frac{n-1}{2}\times\frac{n-2}{3}x^3, \text{\&c.}\\
* + nx - n\times\frac{n-2}{1}\times x^2 + n\times\frac{n-2}{1}\times\frac{n-3}{2}x^3, \text{\&c.}\\
* \quad * + n\times\frac{n-3}{2}\times x^2 - n\times\frac{n-3}{2}\times\frac{n-4}{1}x^3, \text{\&c.}\\
* \quad * \quad * \quad + n\times\frac{n-4}{2}\times\frac{n-5}{3}x^3, \text{\&c.}\\
\hfill \text{\&c.}\\
\hfill +\overline{2+b}\times x^m
\end{array}\right\} = Pa^2\times Pb^2, \text{\&c.}$$

Which contracted, by adding together the homologous Terms, becomes $1 * * *$, &c. Hence it appears, that the Coefficients do every where deſtroy one another, except in the firſt, laſt, and the middlemoſt of the ſaid Terms; and that the middle Term would likewiſe vaniſh, if inſtead of $\overline{2+2b} \times x^{m}$, the correſponding Term of the above Series $\overline{1 \backsim x}|^{n} + nx \times \overline{1 \backsim x}|^{u-2}$, or that where the Exponent of x is m, was to be added; wherefore this Term being $n \times \frac{\frac{1}{2}n-1}{2} \times \frac{\frac{1}{2}n-2}{3} \ldots\ldots \frac{1}{\frac{1}{2}n}$ into x^{m} $(=2x^{m})$ as is eaſy to perceive from the Law of Continuation, we have $1+2bx^{m}+x^{2m}=Pa^{2} \times Pb^{2} \times Pc^{2}$, &c. or, $AO^{2m} + 2Ok \times AO^{m-1} \times PO^{m} + PO^{2m} = Pa^{2} \times Pb^{2}$, &c. And, when the Point k is taken on the other Side of O, Ok becoming $-Ok$, $AO^{2m} - 2Ok \times AO^{m-1} \times PO^{m} + PO^{2m}$ will be equal to $Pa^{2} \times Pb^{2} \times Pc^{2}$, &c.

Q. E. D.

C O R O L. I.

IF C be taken at B; then will $Ok = AO$, and $Pa^{2} \times Pb^{2} \times Pc^{2}$, &c. $= AO^{2m} + 2AO^{m} \times PO^{m} + PO^{2m}$; where, by taking the Square Root on each Side, we have $Pa \times Pb \times Pc$, &c. $= AO^{m} + PO^{m}$.

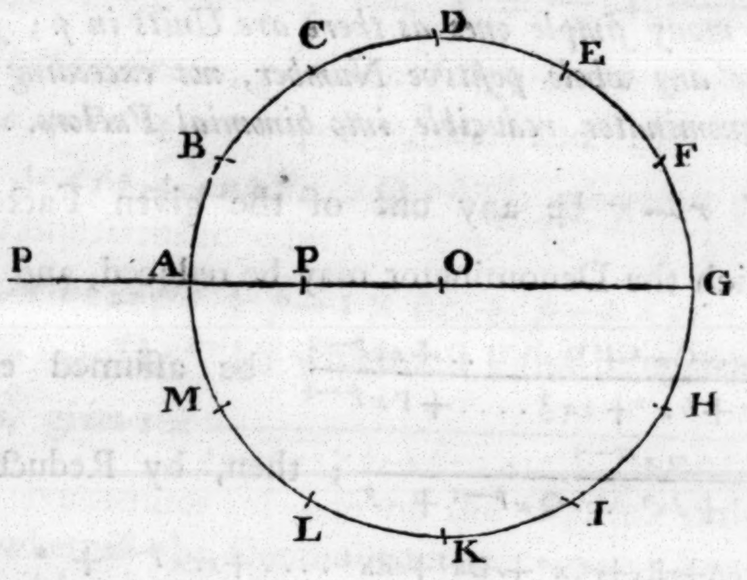
C O R O L. II.

BUT if C comes into A; then A being $=0$, and $Ok = AO$, $AO^{2m} - 2Ok \times AO^{m-1} \times PO^{2m} = Pa^{2} \times Pb^{2}$, &c. will therefore become $AO^{2m} - 2AO^{m} \times$

$PO^{m} + PO^{2m} = Pa^{2} \times Pb^{2} \times Pc^{2}$, &c. And $Pa \times Pb$, &c. $= AO^{m} \backsim PO^{m}$.

COROL. III.

HENCE it is manifeſt, that if any Circle ABCD, &c. be divided into as many equal Parts as there are Units in $2m$ (m being any whole Number whatſoever) and if in the Radius OA, produced thro' A, any one of the Points of Diviſion, a Point as P be aſſumed any where, either within or without the Circle, $PA \times PC \times PE \times PG$, &c. will be $= AO^{m} \backsim PO^{m}$, $PB \times PD \times PF \times PH$, &c. $= AO^{m} + PO^{m}$, and $PA \times PB \times PC \times PD \times PE$, &c. $= AO^{2m} \backsim PO^{2m}$.



Of the Reduction of Compound Fractions into more ſimple ones.

PROPOSITION.

To divide a Compound Fraction, as $\frac{vx^{m-1}}{a+bx+cx^2+dx^3 \dots x^p}$, *into as many ſimple ones as there are Units in* p; *ſuppoſing* m *to be any whole poſitive Number, not exceeding* p, *and the Denominator reducible into binomial Factors.*

LET $r-x$ be any one of the given Factors into which the Denominator may be reduced, and let $\frac{s}{r-x}$ $+ \frac{A+Bx+Cx^2+Dx^3 \dots +tx^{p-2}}{f+gx+hx^2+ix^3 \dots +Px^{p-1}}$ be aſſumed equal to $\frac{vx^{m-1}}{a+bx+cx^2+dx^3 \dots Qx^{p-1}+x^p}$; then, by Reduction, we have

$$\left\{\begin{array}{l} rA+rBx+rCx^2+rDx^3+rEx^4 \dots +rtx^{p-2}+ * \\ * -Ax-Bx^2-Cx^3-Dx^4 \dots\dots\dots\dots -tx^{p-1} \\ sf+sgx+shx^2+six^3+skx^4 \dots\dots\dots\dots +sPx^{p-1} \\ \qquad\qquad\qquad\qquad\qquad -vx^{m-1} \end{array}\right\} = 0,$$

because $r-x \times \overline{f+gx+hx^2}$, &c. $= a+bx+cx^2$, &c. Hence, by comparing the homologous Terms, we get $A = \frac{-fs}{r}$, $B = \frac{-fs}{r^2} - \frac{gs}{r}$, $C = \frac{-fs}{r^3} - \frac{gs}{r^2} - \frac{hs}{r}$, &c. laſtly t, or $sP = \frac{-fs}{r^{p-1}} - \frac{gs}{r^{p-2}} - \frac{hs}{r^{p-3}}$, &c. $+ \frac{vr^{m-1}}{r^{p-1}}$; wherefore $fs+grs+hsr^2+isr^3+ksr^4 \dots + sPr^{p-1} = vr^{m-1}$ and $s = \frac{vr^m}{fr+gr^2+hr^3+ir^4+kr^5 \dots +Pr^p}$:

But, becauſe $\overline{r-x}\times\overline{f+gx+h^2x+ix^3\ldots+Px^{p-1}}$ is $=a+bx+cx^2+dx^3\ldots\ldots Qx^{p-1}+x^p$ we have

$$\left.\begin{array}{l} rf+rgx+rhx^2+rix^3+rkx^4\ldots\ldots rPx^{p-1}\ * \\ *\ -fx-gx^2-hx^3-ix^4\ldots\ldots\ldots\ldots-Px^p \\ -a-bx-cx^2-dx^3-ex^4\ldots\ldots Qx^{p-1}-x^p \end{array}\right\}=0.$$

and therefore $f=\frac{a}{r}$, $g=\frac{a}{r^2}+\frac{b}{r}$, $h=\frac{a}{r^3}+\frac{b}{r^2}+\frac{c}{r}$, $i=\frac{a}{r^4}+\frac{b}{r^3}+\frac{c}{r^2}+\frac{d}{r}$, and $P=\frac{a}{r^p}+\frac{b}{r^{p-1}}+\frac{c}{r^{p-2}}$ &c. Whence $fr=a$, $gr^2=a+br$, $hr^3=a+br+cr^2$, $ir^4=a+br+cr^2+dr^3$, &c. and ∴ $Pr^p=a+br+cr^2+dr^3+er^4\ldots\ldots Qr^{p-1}$; wherefore (by adding all theſe Equations together) there will be $fr+gr^2+hr^3\ldots\ldots+Pr^p=pa+\overline{p-1}\times br+\overline{p-2}\times cr^2+\overline{p-3}\times dr^4\ldots\ldots Qr^{p-1}$; which laſt Value being ſubſtituted in that of s, gives $s=\frac{vr^m}{pa+\overline{p-1}\times br+\overline{p-2}\times cr^2+\overline{p-3}\times dr^3\ldots\ldots Qr^{p-1}}$ for the Numerator of one of the required ſimple Fractions; whereof the Denominator is $r-x$; from whence, by Inſpection, the Numerators anſwering to the other given Factors or Denominators are eaſily obtained: For, if $R-x$, $S-x$, $T-x$, &c. be the ſaid Factors into which $a+bx+cx^2+dx^3\ldots\ldots\ldots Qx^{p-1}+x^p$ is reducible, or $\overline{R-x}\times\overline{S-x}\times\overline{T-x}$, &c. be $=a+bx+cx^2$, &c. And $\frac{1}{pa+\overline{p-1}\times bR+\overline{p-2}\times cR^2+\overline{p-3}\times dR^3\ldots QR^{p-1}}$ be put $=A$, $\frac{1}{pa+\overline{p-1}\times bS+\overline{p-2}\times cS^2+\overline{p-3}\times dS^3\ldots\ldots QS^{p-1}}=B$,

&c. &c. it is evident, that $\frac{vx^{m-1}}{a+bx+cx^2+dx^3\ldots\ldots x^p}$ will be equal to $\frac{AvR^m}{R-x}+\frac{BvS^m}{S-x}+\frac{CvT^m}{T-x}$, *&c.* *Q. E. I.*

EXAMPLE I.

LET the Fraction $\frac{x^{2-1}}{2-3x+x^2}$, $=\frac{x}{\overline{1-x}\times\overline{2-x}}$, be propoſed. Then will $a=2$, $b=-3$, $c=1$, $d=0$, $e=0$, *&c.* $v=1$, $p=2$, $m=2$, $R=1$, $S=2$, $T=0$, *&c.* $A=1$, $B=-\frac{1}{2}$, and therefore $\frac{x}{2-3x+x^2}=\frac{1}{1-x}-\frac{2}{2-x}$.

EXAMPLE II.

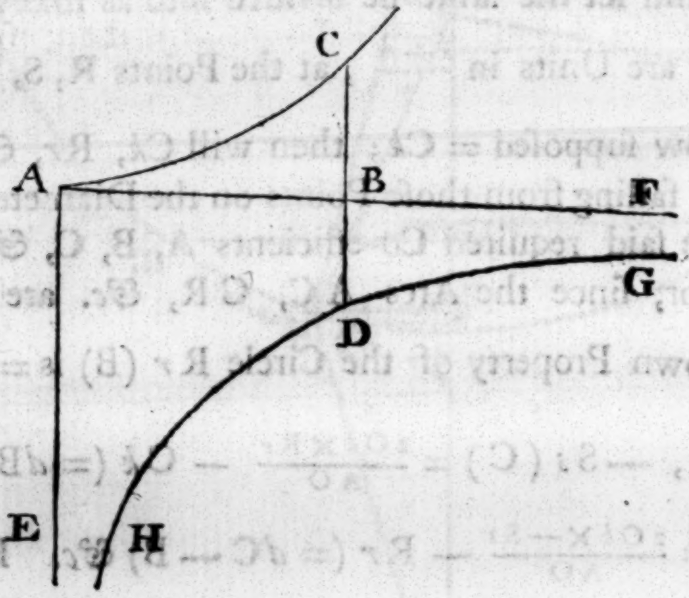
IF the given Fraction be $\frac{vx^{m-1}}{1\pm2\acute{b}x^n+x^{2n}}$; then, by comparing $a+bx+cx^2+dx^3\ldots\ldots Qx^{p-1}+x^p$ with $1\pm2\acute{b}x^n+x^{2n}$, we have $a=1$, $p=2n$, the Coefficient of the middle Term $=\pm2\acute{b}$, and all the reſt except the laſt $=0$; wherefore A $\left(=\frac{1}{pa+\overline{p-1}\times br+\overline{p-2}\times cr^2\ldots Qr^{p-1}}\right)$ will in this Caſe become $=\frac{1}{2n\times\overline{1\pm\acute{b}r^n}}$, $B=\frac{1}{2n\times\overline{1\pm bs^n}}$, *&c.*

A General Quadrature of Hyperbolical Curves.

PROPOSITION

There are two Curves A C, H D G, *having the ſame common Abſciſſa* A B (x) *whoſe Ordinates* B C *and* B D *are* $\frac{x^{n+r-1}}{1 \pm dx^n + x^{2n}}$, and $\frac{x^{n-r-1}}{1 \pm dx^n + x^{2n}}$; *To find the Area of each; ſuppoſing r and n to be any whole poſitive Numbers, and the Denominator* $1 \pm dx^n + x^{2n}$ *not reducible into two binomial Factors.*

LET n be taken in r as often as poſſible, and the Remainder be denoted by m; and let $Ax^{r-n} + Bx^{r-n2} + Cx^{r-3n} \ldots kx^{2n+m} + lx^{n+m} + sx^m + \frac{tx^{n+m} + vx^m}{x^{2n} - dx^n + 1}$



be aſſumed $= \frac{cx^{n-r}}{x^{2n} - dx^n + 1}$; c being any given Quantity:

I i

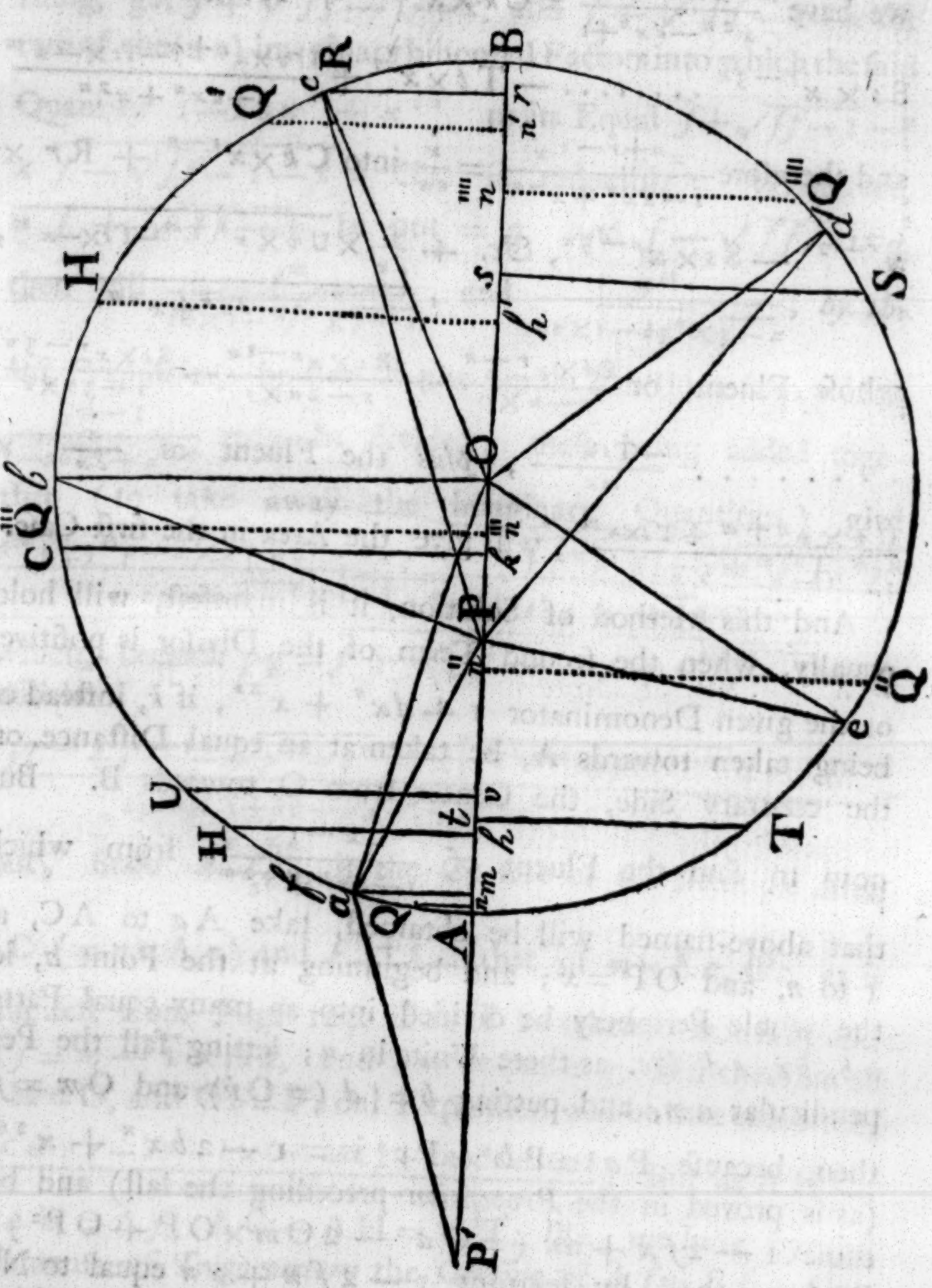
Then, by reducing this Equation into one Denomination, we ſhall have $\left\{\begin{matrix}+A\\-c\end{matrix}\right\}x^{r+n}+$

$$\left\{\begin{matrix}+B\\-dA\end{matrix}\right\}x^{r}+\left\{\begin{matrix}+C\\-dB\\+A\end{matrix}\right\}x^{r-n}+\left\{\begin{matrix}+D\\-dC\\+B\end{matrix}\right\}x^{r-2n}\ldots\ldots+$$

$$\left\{\begin{matrix}+s\\-dl\\+k\end{matrix}\right\}x^{2n+m}+\left\{\begin{matrix}*\\-ds\\+l\\+t\end{matrix}\right\}x^{n+m}+\left\{\begin{matrix}*\\*\\s\\v\end{matrix}\right\}x^{m}=0\ ;$$ and

therefore $A=c$, $B=dA$, $C=dB-A$, $D=dC-B$, $E=dD-C$ .... $t=ds-l$, $v=-s$.

But, now in order to conſtruct theſe ſeveral Co-efficients; with the Radius 1, and Centre O, *Fig.* 2. let the Circle AB be deſcribed; take $Ok=\frac{1}{2}d$, Ck perpendicular to AB, meeting the Circle in C, and the Arch CBU to the Arch AC, as $r-m$ to n; and let the ſame be divided into as many equal Parts as there are Units in $\frac{r-m}{n}$, at the Points R, S, T, &c. and let c be now ſuppoſed $=Ck$; then will Ck, Rr, &c. the Perpendiculars falling from thoſe Points on the Diameter AB, be equal to the ſaid required Co-efficients A, B, C, &c. reſpectively: For, ſince the Arcs AC, CR, &c. are equal, by a well known Property of the Circle Rr (B) is $=\frac{2Ok}{AO}\times Ck\ (=dA)$, $-Ss\ (C)=\frac{2Ok\times Rr}{AO}-Ck\ (=dB-A)$, $-Tt\ (D)=\frac{2Ok\times -Ss}{AO}-Rr\ (=dC-B)$ &c. Hence,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we have $\frac{cx^{n+r}}{x^{2n}-dx^n+1} = Ck \times x^{r-n} + Rr \times x^{r-2n} -$ $Ss \times x^{r-3n} \ldots\ldots\ldots - Tt \times x^m + \frac{Uv \times x^{n+m} - Tt \times -x^m}{1-dx^n+x^{2n}}$ and therefore $\frac{x^{n+r-1}\dot{x}}{1-dx^n+x^{n2}} = \frac{\dot{x}}{cx}$ into $Ck \times x^{r-n} + Rr \times$ $x^{r-2n} - Ss \times x^{r-3n}$, &c. $+ \frac{\frac{\dot{x}}{cx} \times \overline{Uv \times x^{n+m} - Tt \times -x^m}}{1-dx^n+x^{2n}}$;

whoſe Fluent, or $\frac{Ck \times x^{r-n}}{r-n \times c} + \frac{Rr \times x^{r-2n}}{r-2n \times c} - \frac{Ss \times x^{r-3n}}{r-3n \times c}$ $\ldots\ldots\ldots \frac{-Tt \times x^m}{mc}$, *plus* the Fluent of $\frac{\dot{x}}{cx} \times$ $\frac{Uv \times x^{n+m} + Tt \times x^m}{1-dx^n \times x^{2n}}$, will give the Area in the firſt Caſe.

And this Method of Solution, it is manifeſt, will hold equally, when the ſecond Term of the Diviſor is poſitive, or the given Denominator $1 + dx^n + x^{2n}$, if k, inſtead of being taken towards A, be taken at an equal Diſtance on the contrary Side, the Center from O towards B. But now to find the Fluent of $\frac{x^{m-1}\dot{x}}{1-dx^n+x^{2n}}$, from which that above-named will be obtained, take Aa to AC, as 1 to n, and $OP = x$; and beginning at the Point a, let the whole Periphery be divided into as many equal Parts, ab, bc, cd, &c. as there Units in n; letting fall the Perpendicular am, and putting $b = \frac{1}{2}d$ ($= Ok$) and $Om = f$; then, becauſe $Pa^2 \times Pb^2 \times Pc^2$ is $= 1 - 2bx^n + x^{2n}$, (as is proved in the *Propoſition* preceding the laſt) and becauſe $1 - 2fx + xx$ ($= Oa^2 - 2Om \times OP + OP^2$) is Pa^2, we ſhall, by feigning $1 - 2fx + xx$ equal to Nothing,

thing, get $\overline{f+\sqrt{ff-1}-x}$, and $\overline{f-\sqrt{ff-1}-x}$ for two of the (2 n) imaginary binomial Factors into which the ſaid Quantity $1-2bx^n+x^{2n}$, or its Equal $\overline{f+\sqrt{ff-1}-x}\times\overline{f-\sqrt{ff-1}-x}\times$ &c. is reducible: Wherefore, if $f+\sqrt{ff-1}$ be put $=p$, and $f-\sqrt{ff-1}=q$, then will $\frac{p^m}{2n\times\overline{1-bp^n}\times\overline{p-x}}$, and $\frac{q^m}{2n\times\overline{1-bq^n}\times\overline{q-x}}$, *by the laſt Propoſition*, be two of the ſimple Fractions into which $\frac{x^{m-1}}{1-2bx^n+x^{2n}}$ may be divided; theſe being added together (to take away the imaginary Quantities) give

$$\frac{p^mq+q^mp-x\times\overline{p^m+q^m}-b\times\overline{p^mq^{n+1}+q^mp^{n+1}}+bx\times\overline{p^mq^n+q^mp^n}}{2n \text{ into } \overline{1-b}\times\overline{p^n+q^n}+bbp^nq^n \text{ into } \overline{p-x}\times\overline{q-x}}$$

which, becauſe $pq=\overline{f+\sqrt{ff-1}}\times\overline{f+\sqrt{ff-1}}$ is $=1$, will be

$$\frac{p^{m-1}+q^{m-1}-x\times\overline{p^m+q^m}-b\times\overline{q^{n+1-m}+p^{n+1-m}}+bx,\ \&c.}{2n \text{ into } 1+bb-b\times\overline{p^n+q^n} \text{ into } 1-2fx+xx}:$$

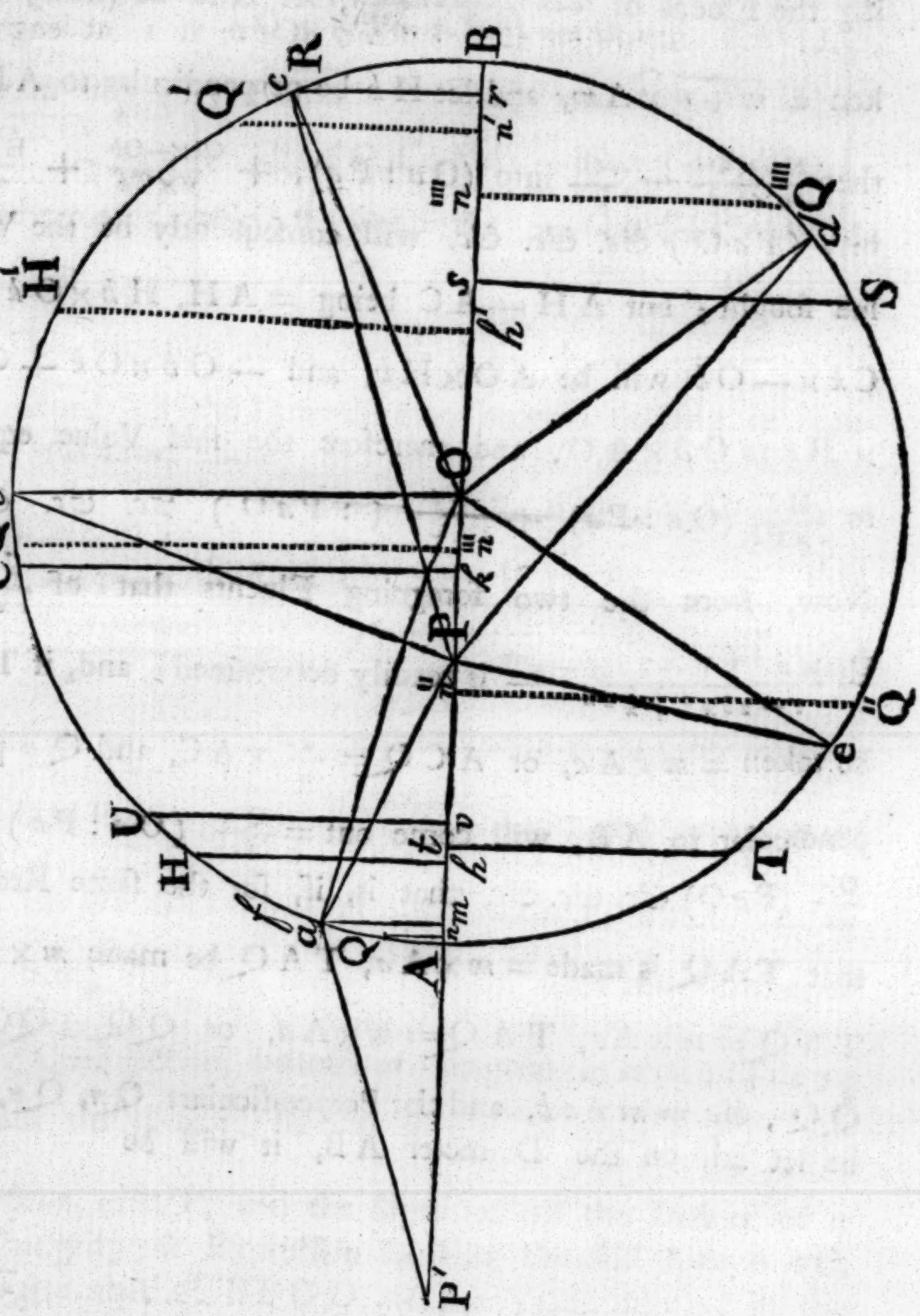
But, ſince $\frac{p^n+q^n}{2}$ is the Co-ſine of the Multiple Arch AC $(=n\times \text{A}a)$ and $\frac{p^m+q^m}{2}$ that of $m\times \text{A}a$, &c. as is manifeſt from *Page* 109. if AH be taken $=m\times \text{A}a$, and $\text{A}f=\overline{m-1}\times \text{A}a$, and am be put $=g$, Hb the Sine of AH $=$ G, and O$b=$ F; our Expreſſion will be thus exhibited,

$$\frac{\text{Coſ. of A}f-x\text{F}-b\times\text{Coſ. of C}f+bx\times\text{Coſ. CH}}{ncc\times 1-2fx+xx}.$$

But as AH — Aa is $=$ Af, AC — AH $=$ CH, &c. we have by the Elements of *Trigonometry* the Co-ſine of $\text{A}f=\text{F}f+\text{G}g$, Coſ. of CH $=b\text{F}+c\text{G}$, and the Coſ. of $\text{C}f=bg\text{G}+bf\text{F}+cf\text{G}-cg\text{F}$; and therefore our Expreſſion will ſtand thus,

$\frac{Ff + Gg - b^2 fF - b^2 gG - bcfG + bcgF + \overline{b^2 F - F + bcG} \times x}{ncc \times 1 - 2fx + xx}$; where, by ſubſtituting $1 - cc$ inſtead of bb, it is, at length, reduced to $\frac{cfF + bgF + cgG - bfG + x \times \overline{bG - cF}}{cn \times \overline{1 - 2fx + xx}}$: And this drawn into $\dot{x}$, is one of the n rational Fractions, (whoſe Denominators are $cn \times Pa^2$, $cn \times Pb^2$, $cn \times Pc^2$, &c.) into which the Quantity $\frac{x^{m-1}\dot{x}}{1 - dx^n + x^{2n}}$, whoſe Fluent we are ſeeking, may be divided. Now, therefore, the Fluent of $\frac{\dot{x}}{1 - 2fx + xx}$ being $\frac{1}{g}$ (PaO) or, $\frac{1}{g}$ into the Arch meaſuring the Angle PaO, and that of $\frac{x\dot{x}}{1 - 2fx + xx}$, equal to the ſame Arch into $\frac{f}{g}$, *plus* $(AO : Pa)$ or the Hyperbolical Logarithm of $\frac{Pa}{AO}$; the Fluent of

$\frac{\overline{cfF + bgF + cgG - bfG} \times \dot{x} + \overline{bG - cF} \times x\dot{x}}{cn \times 1 - 2fx + xx}$, that of thoſe Fractions whoſe Denominator is $cn \times Pa^2$) will be $= \overline{\frac{Gb}{cn} - \frac{F}{n}} \times (Oa : Pa) + \overline{\frac{bF}{nc} + \frac{G}{n}} \times (PaO)$, Or, $= \frac{Hb \times Ok}{n \times Ck} - \frac{Ob}{n}$ into $(Oa : Pa)$, $+ \frac{Ob \times Ok}{n \times Ck} + \frac{Hb}{n}$ into (PaO): From whence the Fluents of the reſt of the Fractions, which make up the required Value, whoſe Denominators are $nc \times Pb^2$, $nc \times Pd^2$, &c. are determined, by Inſpection, ſince the Manner of Conſtruction muſt neceſſarily be the ſame in all of them. Next, from hence to find the Fluent of $\frac{x^{n+m-1}\dot{x}}{1 - 2bx^n + x^{2n}}$: For the very ſame Reaſon that AH was taken $= m \times Aa$ in finding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ing the Fluent of $\frac{x^{m-1}\dot{x}}{1-2bx^{n}+x^{2n}}$, let AH' be (now) taken $=\overline{m+n}\times Am$, and let $H'b'$ be perpendicular to AB; then $\frac{H'b'\times Ok}{n\times Ck}-\frac{-Ob'}{n}$ into $(Oa:Pa)$, $+\frac{Ok\times -Ob}{n\times Ck}+\frac{H'b'}{n}$ into (PaO) &c. &c. &c. will conſequently be the Value ſought; but $AH'-AC$ being $=AH$, $H'b'\times Ok-Ck\times -Ob'$ will be $AO\times Hb$, and $-Ob'\times Ok+Ck\times H'b'=Ob\times AO$, and therefore the ſaid Value equal to $\frac{Hb}{n\times Ck}(Oa:Pa)+\frac{Ob}{n\times Ck}(:PaO)$ &c. &c. &c.

Now, from the two foregoing Fluents that of $\frac{\dot{x}}{cx}\times\frac{Uv\times x^{n+m}-Tt\times -x^{m}}{1-2bx^{n}+x^{2n}}$ is readily determined; and, if TQ be taken $=m\times Aa$, or $ACQ=\frac{r}{n}\times AC$, and Qn perpendicular to AB, will come out $=\frac{Qn}{nc}(Oa:Pa)+\frac{On}{nc}(PaO)$ &c. &c. &c. that is, if, for the ſame Reaſon that TAQ is made $=m\times Aa$, TAQ' be made $m\times Ab$, $TAQ''=m\times Ac$, $TAQ'''=m\times Ad$, or $QQ'=Q'Q''=Q''Q'''$, &c $=m\times ab$, and the Perpendiculars $Q'n'$, $Q''n''$, &c. be let fall on the Diameter AB, it will be

$\frac{1}{n\times Ck}$

$$\frac{1}{n\times Ck}\text{ into }\left\{\begin{array}{l} Qn\,(Oa : Pa) + On\,(\ PaO) \\ Q'n'\,(Ob : Pb) - On'\,(\ PbO) \\ -Q''n''\,(Oc : Pc) + On''\,(\ PcO) \\ Q'''n'''\,(Od : Pd) + On'''\,(-PdO) \\ -Q''''n''''\,(Oe : Pe) - On''''\,(-PeO) \\ \qquad\&c. \qquad\qquad\qquad\qquad \&c. \end{array}\right\}$$

This therefore added to $\frac{Ck\times x^{r-n}}{r-n\times c}+\frac{Rr\times x^{r-2n}}{r-2n\times c}$, &c. continued 'till the Denominators become nothing or negative, (as above found) will be the required Area in the firſt Caſe. But for the Area in the other, where the Ordinate is $\frac{x^{n-r-1}}{1\mp dx^{n}+x^{2n}}$, let x be put equal to $\frac{1}{y}$, or $y=\frac{1}{x}$; then will $\frac{x^{r+n+1}}{1\mp dx^{n}+x^{2n}}$ be $=\frac{-y^{+n+r-1}}{1\mp dy^{n}+y^{2n}}$, and $\dot{x}=\frac{-\dot{y}}{yy}$; and therefore $\frac{x^{n-r-1}\dot{x}}{1\mp dx^{n}+x^{2n}}$, or the Fluxion of the propoſed Area ABDHEA, equal to $\frac{-y^{n+r-1}\dot{y}}{1\mp dy^{n}+y^{2n}}$, and conſequently that of BFGD equal to $\frac{y^{n+r-1}\dot{v}}{1\mp dy^{n}+y^{2n}}$; which Expreſſion being the very ſame in Form with the Fluxion of the Area ABC; it is manifeſt, that if P, in the foregoing Conſtruction, inſtead of being taken at the Diſtance x from the Centre, be put at the Diſtance $\frac{1}{x}$ $(=y)$ therefrom, as at P', and the Signs of all the Indices of x be changed, the Expreſſion ſhewing the ſaid Area ABC, will give that of BFGD, or the Value required in this

Caſe

Caſe: Which therefore is $\frac{Ck}{r-n}+\frac{Rr\times x^{n}}{r-2n}-\frac{Ss\times x^{2n}}{r-3n}$, &c. into $\frac{x^{n-r}}{Ck}$, *plus* $\frac{1}{n\times Ck}$ into $Qn\ (Oa : \acute{P}a) + On\ (\acute{P}aO)$ &c. &c. But OP (x) being to Oa (1), as Oa to O$\acute{P}$ $\left(\frac{1}{x}\right)$, the Triangles OPa, and O$a\acute{P}$, will be ſimilar, and therefore the Angle $\acute{P}aO = OPa$, and $Oa : \acute{P}a$, $OP : Pa$, &c. wherefore the ſaid Area will be $\frac{Ck}{r-n}+\frac{Rr\times x^{n}}{r-2n}-\frac{Ss\times x^{2n}}{r-3n}$, &c. into $\frac{x^{n-r}}{Ck}$

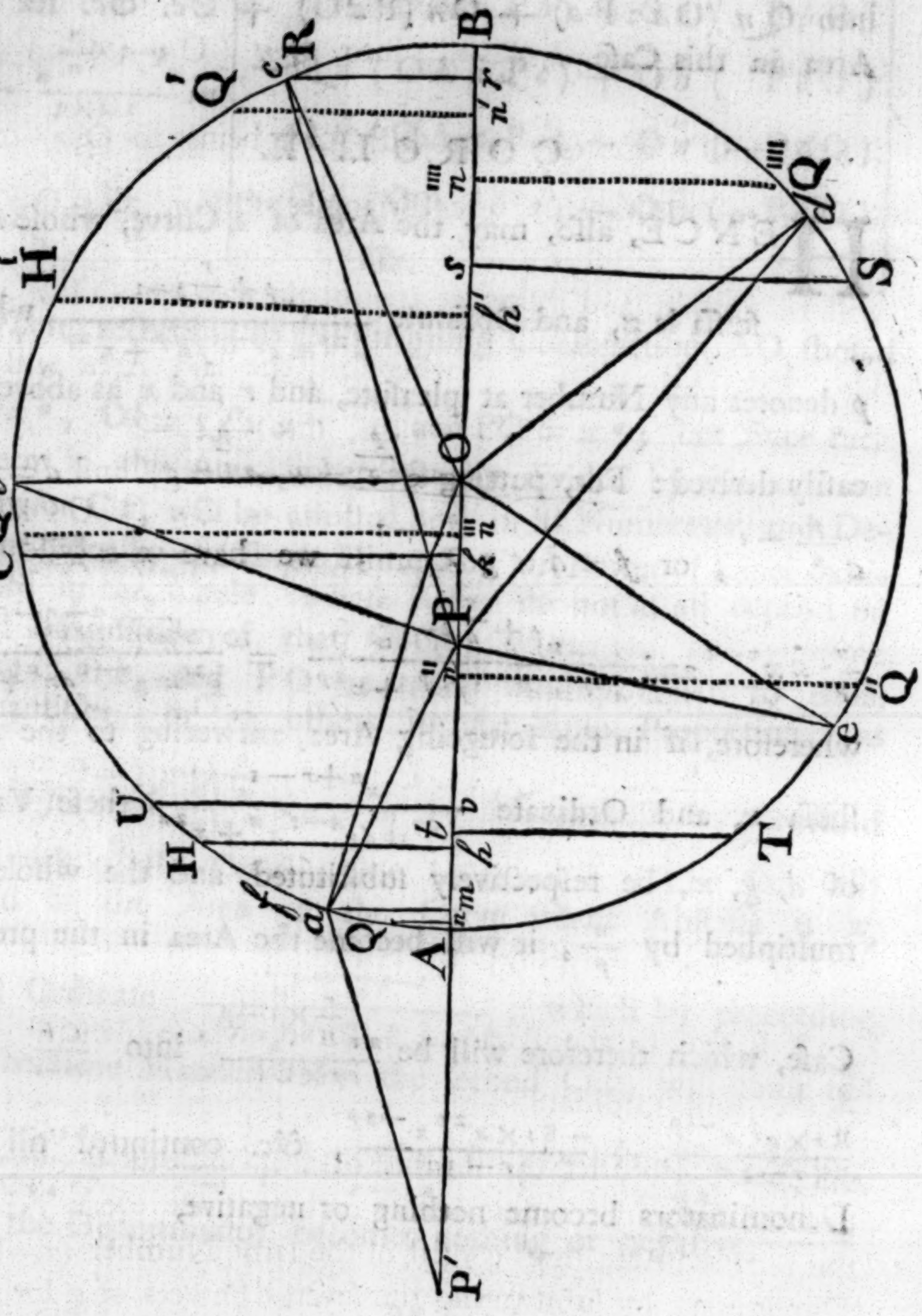
$$plus\ \frac{1}{n\times Ck}\ \text{into}\left\{\begin{array}{l} Qn\,(OP : Pa) + On\,(OPa) \\ Q'n'\,(OP : Pb) - O'n'\,(OPb) \\ -Q''n''\,(OP : Pc) + O''n''\,(OPc) \\ Q'''n'''\,(OP : Pd) + O'''n'''\,(-OPd) \\ -Q''''n''''\,(OP : Pe) - O''''n''''\,(-OPe) \\ \quad\&c. \qquad\qquad\qquad \&c. \end{array}\right]$$

Q. E. I.

C O R O L. I.

HENCE the Area of a Curve, whoſe Abſciſſa is x, and Ordinate $\frac{x^{n+r-1}}{g^{2n}\pm g^{n-1}dx^{n}+x^{2n}}$ may be eaſily obtained: For, let the Radius of the Circle AB, now be denoted by g, the reſt as before; then, ſince every Term in the required Area muſt conſiſt of the ſame Number ($r-n$) of Dimenſions, by ſubſtituting the ſeveral Powers of g for thoſe of AO, or Unity in our former Area, it will become

$\frac{x^{r-n}}{Ck}$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$\frac{x^{\overline{r-n}}}{Ck}$ into $\frac{Ck}{r-n} + \frac{Rr \times g^n x^{\overline{-n}}}{r-2n} - \frac{Ss \times g^{2n} x^{\overline{-2n}}}{r-3n}$, &c. $+ \frac{g^{\overline{r-n}}}{n \times Ck}$ into $Qn\ (Oa : Pa) + On\ (PaO)$ + &c. &c. for the Area in this Caſe.

COROL. II.

HENCE, alſo, may the Area of a Curve, whoſe Abſciſſa is z, and Ordinate $\frac{z^{p + \frac{r}{n}p - 1}}{a^{2p} \pm a^{p-1} f z^p + z^{2p}}$, where p denotes any Number at pleaſure, and r and n as above, be eaſily derived: For, putting $z^{\frac{p}{n}} = x$, $a^{\frac{p}{n}} = g$, and $d = f \times a^{\frac{p}{n} - 1}$, or $f = d \times g^{\frac{n}{p} - 1}$, we have $\dot{z} z^{p-1} = \frac{n}{p} \dot{x} x^{n-1}$ and $\frac{z^{p + \frac{r}{n}p - 1} \dot{z}}{a^{2p} \pm a^{p-1} f z^p + z^{2p}} = \frac{\frac{n}{p} x^{n+r-1} \dot{x}}{g^{2n} \pm g^{n-1} d x^n + x^{2n}}$ wherefore, if in the foregoing Area, anſwering to the Abſciſſa x, and Ordinate $\frac{x^{n+r-1}}{g^{2n} \pm d g^{n-1} x^n + x^{2n}}$, theſe Values of d, g, x, be reſpectively ſubſtituted, and the whole be multiplied by $\frac{n}{p}$, it will become the Area in the preſent Caſe, which therefore will be $\frac{n z^{\frac{r}{n} \times p - p}}{p \times Ck}$ into $\frac{Ck}{r-n} + \frac{Rr \times a^p z^{-p}}{r-2n} + \frac{-Ss \times a^{2p} z^{-2p}}{r-3n}$, &c. continued 'till the Denominators become nothing or negative,

+

$$+\ \frac{a^{\frac{r}{n}\times p-p}}{p\times Ck}\ in\left\{\begin{array}{llll} & Qn\,(OA:Pa) & +\,On & (\ \ PaO)\\ & \overset{\prime}{Q}\overset{\prime}{n}\,(OA:Pb) & -\,O\overset{\prime}{n} & (\ \ PbO)\\ - & \overset{\prime\prime}{Q}\overset{\prime\prime}{n}\,(OA:Pc) & +\,O\overset{\prime\prime}{n} & (\ \ PcO)\\ & \overset{\prime\prime\prime}{Q}\overset{\prime\prime\prime}{n}\,(OA:Pd) & +\,O\overset{\prime\prime\prime}{n} & (-PdO)\\ - & \overset{\prime\prime\prime\prime}{Q}\overset{\prime\prime\prime\prime}{n}\,(OA:Pe) & -\,O\overset{\prime\prime\prime\prime}{n} & (-PeO)\\ & \quad\&c. & & \quad\&c. \end{array}\right\}$$

Where, according to the foregoing Conſtruction, AO ſhould $= a^{\frac{p}{n}}$, $Ok = \frac{1}{2}f\times a^{\frac{p}{n}-1}$, and $PO = z^{\frac{p}{n}}$; but ſince each Term in the Area, when actually divided by the common Diviſor Ck, will be affected both in its Numerator and Denominator by one ſingle Dimenſion or Power of Lines exhibited in the Circle, whoſe Ratios do not at all depend on the Magnitude of that Circle, it matters not, whether AO, Ok, and PO be taken exactly equal to thoſe Quantities, or to others in the ſame Proportion, as a, $\frac{1}{2}f$, and $\left.\overline{\tfrac{z}{a}}\right|^{\frac{p}{n}}\times a$, or 1, $\frac{\frac{1}{2}f}{a}$, and $\left.\overline{\tfrac{z}{a}}\right|^{\frac{p}{n}}$, provided the reſt of the Conſtruction be retained. The like will hold in the Area of the Curve whoſe Abſciſſa is z, and Ordinate $\frac{z^{p-\frac{r}{n}p-1}}{a^{2p}\pm fa^{p-1}z^{p}+z^{2p}}$, which by proceeding in the ſame Manner, from the ſecond Caſe, will come out

$$\frac{nz^{-\frac{r}{n}p+p}}{p\times Cka^{2p}}\ in\ \frac{Ck}{r-n}+\frac{Rr\times z^{p}a^{-p}}{r-2n}+\frac{-Ss\times z^{2p}a^{-2p}}{r-3n},\ \&c.$$

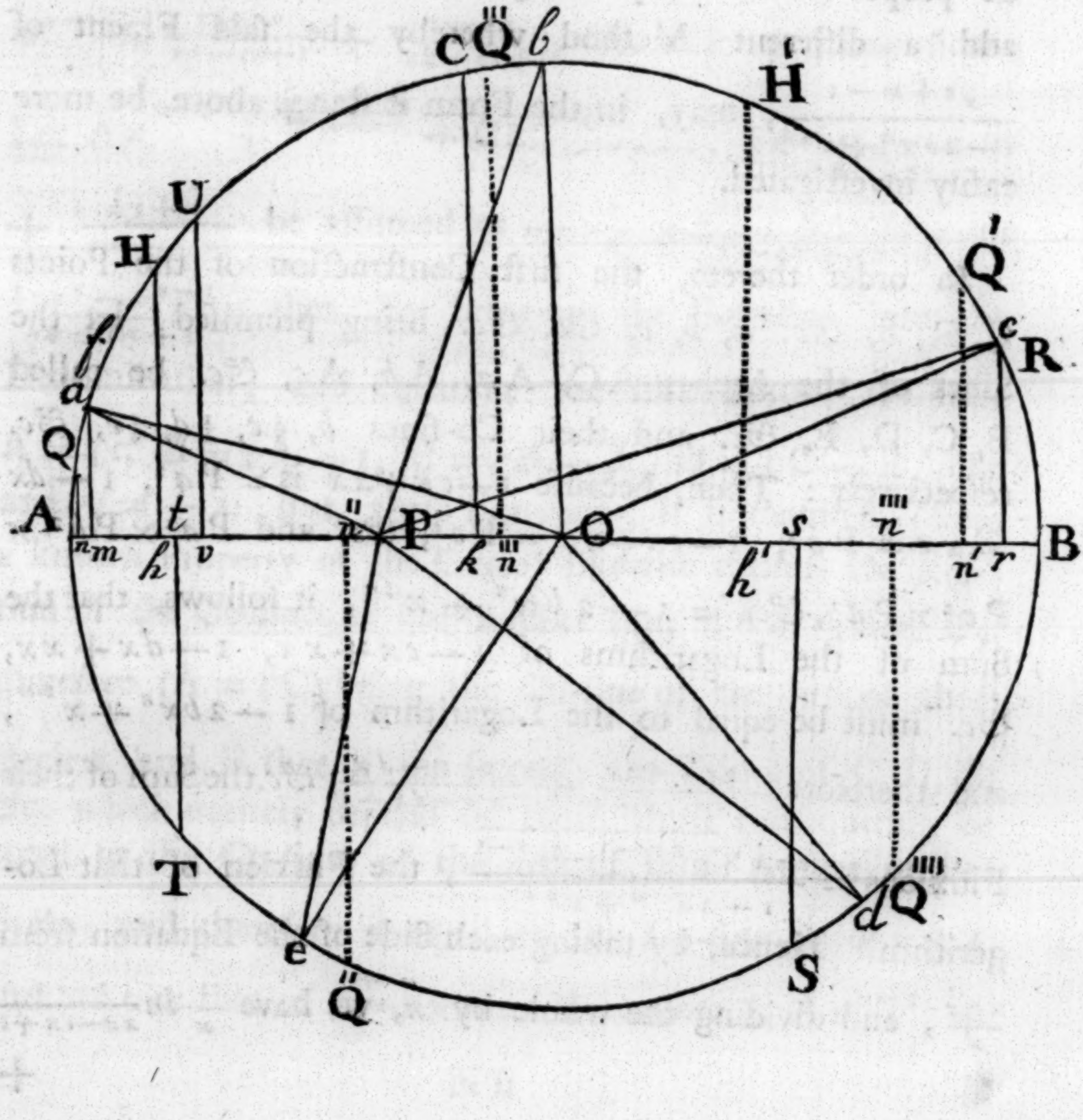
till the Denominator becomes nothing or negative,

$$+\frac{a^{-\frac{r}{n}p-p}}{p\times Ck}\ in\left\{\begin{array}{l} Qn\,(OP:Pa)+On\,(\ aPO)\\ Q'n'\,(OP:Pb)-On'\,(\ bPO)\\ -Q''n''\,(OP:Pc)+On''\,(\ cPO)\\ Q'''n'''\,(OP:Pd)+On'''\,(-dPO)\\ -Q''''n''''\,(OP:Pe)-On''''\,(-ePO)\\ \&c. \qquad\qquad\qquad\qquad \&c.\end{array}\right\}$$

Hence, to find the Area ABCA of a Curve, whoſe Abſciſſa is z, and Ordinate $\frac{z^{p+\frac{r}{n}p-1}}{a^{2p}\pm fa^{p-1}z^{p}+z^{2p}}$, or the Area BFGD of that whoſe Abſciſſa is z (= AB) and Ordinate $\frac{z^{p-\frac{r}{n}p-1}}{a^{2p}\pm fa^{p-1}z^{p}+z^{2p}}$ (ſee the foregoing *Fig.*), we have this Conſtruction. From the Centre O of the Circle ACB, whoſe Radius is = 1, take the Point k in the Diameter AB, towards A or B, according as the Sign of $fa^{p-1}z^{p}$ is − or +, ſo that Ok may = $\frac{f}{2a}$, and draw the Perpendicular Ck to the Point k in that Diameter, meeting the Circle in C; make Aa to AC, as 1 to n, and ACQ to AC, as r to n, and CR, RS, ST, &c. each equal to AC; and, beginning at the Point a, let the whole Circle be divided into as many equal Parts as there are Units in n, as ab, bc, cd, &c. take OP = $\overline{\frac{z}{a}}\Big|^{\frac{p}{n}}$, and each of the Arches QQ′, Q′Q″, &c. = the Arch ab into the Remainder of r divided by n, and draw Pa, Oa, Pb, Ob, &c. and the Perpendiculars Rr, Ss, Tt, &c. Qn, Q′n′, Q″n″, &c. to the Diameter AB;

then will the Areas be reſpectively as above exhibited. And it muſt be obſerved, that this Solution holds in all Caſes where f is leſs than $2a$, and r and n whole poſitive Numbers.

Note. That (OP : Pa) is put to denote the hyperbolical Logarithm of $\frac{Pa}{OP}$, and (aPO) the Meaſure of the Angle aPO in Parts of Radius or Unity; the like is to be underſtood of any other.


C
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$\overset{\prime\prime\prime}{n}$
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s
$\overset{\prime\prime\prime\prime}{n}$
B
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r
$\overset{\prime}{Q}$
c
R
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S
d
$\overset{\prime\prime\prime\prime}{Q}$

SCHOLIUM.

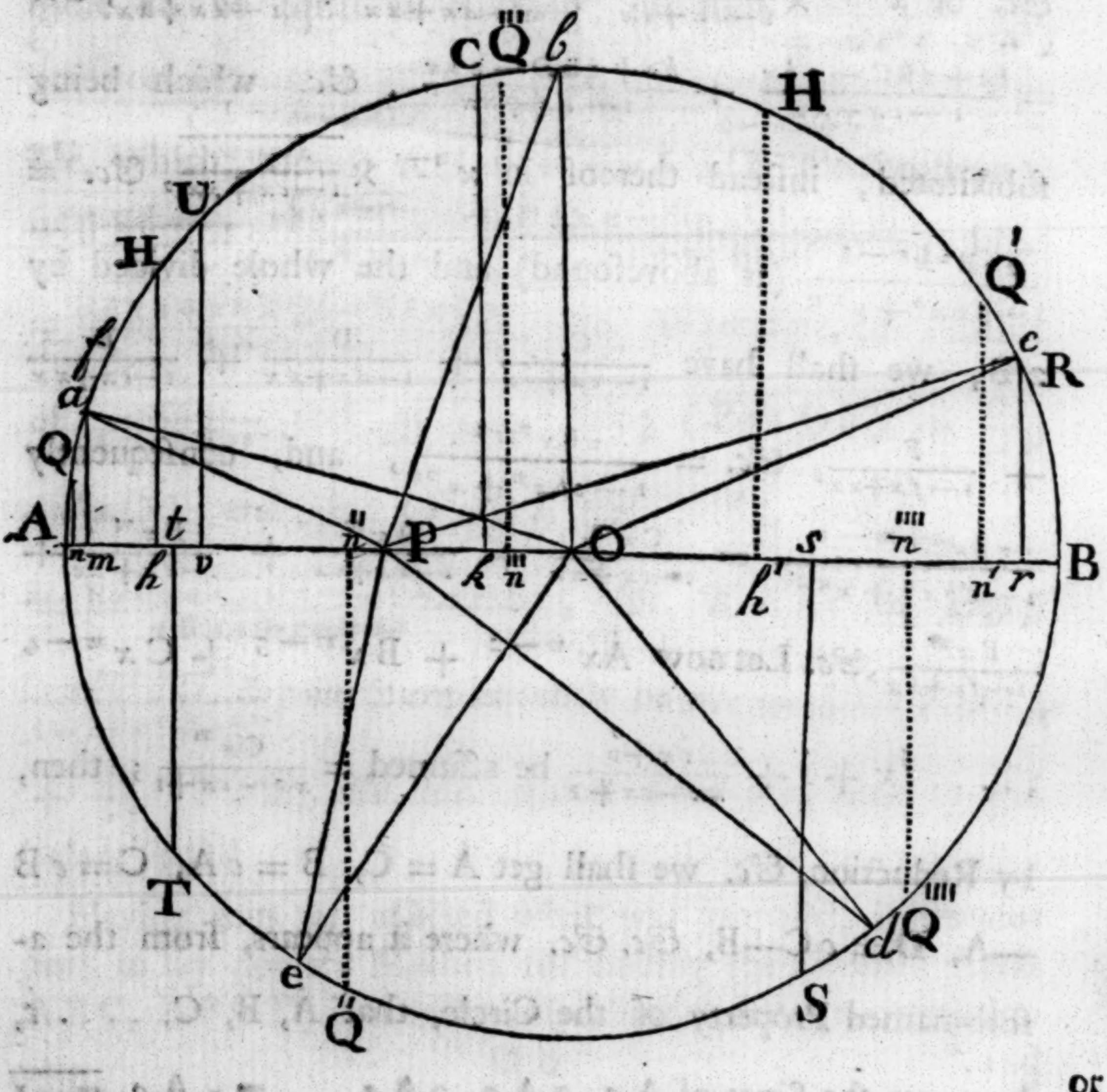
THE above Solution being ſomewhat intricate, others by infinite Series, where they will converge, may be thought preferable; but as the greateſt, if not the only, Difficulty in what has been here delivered, lies in finding the Fluent of $\frac{x^{m-1}\dot{x}}{1-2bx^{n}+x^{2n}}$, or $\frac{x^{n+m-1}}{1-2bx^{n}+x^{2n}}$, it may be proper (before any thing is offered on this Head) to add a different Method whereby the ſaid Fluent of $\frac{x^{n+m-1}}{1-2bx^{n}+x^{nn}}$, may, in the Form it ſtands above, be more eaſily inveſtigated.

In order thereto, the firſt Conſtruction of the Points C, R, S, *&c.* *a*, *b*, *c*, *&c.* *&c.* being premiſed, let the Sines of the Arches AC, A*a*, A*b*, A*c*, *&c.* be called B, C, D, E, *&c.* and their Co-ſines *b*, $\frac{1}{2}c$, $\frac{1}{2}d$, $\frac{1}{2}e$, *&c.* reſpectively: Then, becauſe $1-cx+xx$ is $=\mathrm{P}a^{2}$, $1-dx+xx=\mathrm{P}b^{2}$, $1-ex+xx=\mathrm{P}c^{2}$, *&c.* and $\mathrm{P}a^{2}\times\mathrm{P}b^{2}\times\mathrm{P}c^{2}\times\mathrm{P}d^{2}$, *&c.* $=1-2bx^{n}+x^{2n}$, it follows, that the Sum of the Logarithms of $1-cx+xx$, $1-dx+xx$, *&c.* muſt be equal to the Logarithm of $1-2bx^{n}+x^{2n}$, and therefore $\frac{2x\dot{x}-c\dot{x}}{xx-cx+1}+\frac{2x\dot{x}-d\dot{x}}{xx-dx+1}$, *&c.* the Sum of their Fluxions $=\frac{2n\dot{x}\times\overline{x^{2n-1}-bx^{n-1}}}{1-2bx^{n}+x^{2n}}$, the Fluxion of that Logarithm. Hence, by taking each Side of the Equation from $\frac{2n\dot{x}}{x}$, and dividing the whole by $\dot{x}$, we have $\frac{1}{x}$ *in* $\frac{2-cx}{xx-cx+1}$

+

$+ \frac{2 - dx}{xx - dx + 1}$, &c. $= \frac{2n \times \overline{x^{-1} - bx^{n-1}}}{1 - bx^{n} + x^{2n}}$; this Equation being multiplied by x^{n}, and the former by $\frac{b}{x}$, and the Products added together, there will be b into $\frac{2x - c}{1 - cx + xx} + \frac{2x - d}{1 - dx + xx}$, &c. $+ x^{n-1}$ into $\frac{2 - cx}{1 - cx + xx} + \frac{2 - dx}{1 - dx + xx}$, &c. $\left(= \frac{\overline{1 - bb} \times 2nx^{n-1}}{1 - 2bx^{n} + x^{2n}}\right) = \frac{2nB^{2}x^{n-1}}{1 - 2bx^{n} + x^{2n}}$. Now, to reduce $x^{n-1} \times \frac{2 - cx}{xx - cx + 1} + \frac{2 - dx}{xx - dx + 1}$, &c. to lower Dimenſions; Let $\acute{A}x^{n-2} + \acute{B}x^{n-3} + \acute{C}x^{n-4} \ldots\ldots \acute{k}x^{2} + \acute{l}x + \acute{s} + \frac{\acute{t}x + \acute{v}}{xx - cx + 1}$ be aſſumed $= -\frac{1}{2} \times \frac{x^{n-1} \times \overline{2 - cx}}{1 - cx + xx}$, or $\frac{\frac{1}{2}cx^{n} - x^{n-1}}{xx - cx + 1}$; then, by bringing the Equation into one Denomination, and equating the like Terms, we have $\acute{A} = \frac{1}{2}c$, $\acute{B} = c\acute{A} - 1$, $\acute{C} = c\acute{B} - \acute{A}$, $\acute{D} = c\acute{C} - \acute{B}$, &c. and $v = -s$: But, theſe Equations, it is manifeſt, (from a known Property of the Circle) likewiſe expreſs the Relation of the Co-ſines of the Arches Aa, 2 Aa, 3 Aa, &c. therefore ($\acute{A} = \frac{1}{2}C$) being the Co-ſine of the firſt of thoſe Arches, and $\acute{B}$ that of the ſecond, the Values of $\acute{C}$, $\acute{D}$, $\acute{E}$, &c. which entirely depend on theſe, muſt conſequently be equal to the Co-ſines of the reſt of thoſe Arches reſpectively; and therefore $\acute{t}$ equal to the Co-ſine of $n \times$ Aa, and $-\acute{v} =$ that of $\overline{n - 1} \times$ Aa. Wherefore, if $\overset{\prime\prime}{A}x^{n-2} +$

$\overset{\prime\prime}{B}x^{n-3}+\overset{\prime\prime}{C}x^{n-4}\ldots\overset{\prime\prime}{k}x^{2}+\overset{\prime\prime}{l}x+\overset{\prime\prime}{s}+\frac{\overset{\prime\prime}{t}x+\overset{\prime\prime}{v}}{xx-dx+1}$, be, in like Manner, aſſumed $=-\frac{1}{2}\times\frac{x^{n-1}\times\overline{2-dx}}{xx-dx+1}$, and $\overset{\prime\prime\prime}{A}x^{n-2}+\overset{\prime\prime\prime}{B}x^{n-3}+\overset{\prime\prime\prime}{C}x^{n-4}$, &c. $=-\frac{1}{2}\times\frac{x^{n-1}\times\overline{2-ex}}{xx-ex+1}$ &c. &c. it follows, for the very ſame Reaſon, that $\overset{\prime\prime}{A}$, $\overset{\prime\prime}{B}$, $\overset{\prime\prime}{C}$, $\overset{\prime\prime}{t}$, $-\overset{\prime\prime}{v}$, will be the Co-ſines of the Arches Ab, 2 Ab, 3 Ab, $n\times$Ab, and $\overline{n-1}\times$Ab, and $\overset{\prime\prime\prime}{A}$, $\overset{\prime\prime\prime}{B}$, $\overset{\prime\prime\prime}{C}$, $\overset{\prime\prime\prime}{t}$, $-\overset{\prime\prime\prime}{v}$, thoſe of A$c$, 2 A$c$, 3 A$c$, $n\times$Ac, $n-1\times$Ac. &c. &c. reſpectively; and we ſhall, by adding together the ſaid Equations, have $-\frac{1}{2}\times x^{n-1}\times\overline{\frac{2-cx}{1-cx+xx}+\frac{2-dx}{1-dx+xx}}$ &c. $=x^{n-2}\times\overline{\overset{\prime}{A}+\overset{\prime\prime}{A}+\overset{\prime\prime\prime}{A}}$, &c. $+x^{n-3}\times\overline{\overset{\prime}{B}+\overset{\prime\prime}{B}+\overset{\prime\prime\prime}{B}+\overset{\prime\prime\prime\prime}{B}}$, &c. $+x^{n-4}\times\overline{\overset{\prime}{C}+\overset{\prime\prime}{C}+\overset{\prime\prime\prime}{C}}$, &c. &c. $+\frac{\overset{\prime}{t}x+\overset{\prime}{v}}{1-cx+xx}+\frac{\overset{\prime\prime}{t}x+\overset{\prime\prime}{v}}{1-dx+xx}$, But, $\overset{\prime}{A}$, $\overset{\prime\prime}{A}$, $\overset{\prime\prime\prime}{A}$, &c. being the Co-ſines of the Arches Aa, Ab, Ac, &c. and therefore the Roots of an Equation, expreſſing the Relation of the Co-ſine of an Arch, to that of another Arch n times as great, wherein the ſecond Term is always wanting (*vide* p. 106.) their Sum muſt therefore be equal to nothing, by common *Algebra*; which is evident even by Inſpection, when n is an even Number; for, then every one of the Points a, b, c, &c. above AB having another Point, of the ſame Conſtruction, diametrically oppoſite to it, the Sines, as well as the Co-ſines, anſwering to thoſe Points, muſt be equal and contrary, and therefore deſtroy each other. In like manner $\overset{\prime}{B}$, $\overset{\prime\prime}{B}$, $\overset{\prime\prime\prime}{B}$, &c. being the Co-ſines of 2 A*d*, 2 A*b*, 2 A*c*, &c. or the Roots of ſuch another Equation, they muſt alſo deſtroy one another, &c. &c. Hence our Equation is reduced to $-\frac{1}{2}x^{n-1}\times\overline{\frac{2-cx}{1-cx+xx}+\frac{2-dx}{1-dx+xx}}$, &c. $=\frac{\overset{\prime}{t}x+\overset{\prime}{v}}{1-cx+xx}+\frac{\overset{\prime\prime}{t}x+\overset{\prime\prime}{v}}{1-dx+xx}+\frac{\overset{\prime\prime\prime}{t}x+\overset{\prime\prime\prime}{v}}{1-ex+xx}$, &c. But ſince $\overset{\prime}{t}$ is found to be the Co-ſine of $n\times$ A *a*


C
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T
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d
e
S
$\overset{\prime\prime}{Q}$

or AC, $\overset{\prime\prime}{t}$ that of $n\times Ab$ or ABC, $\overset{\prime\prime\prime}{t}$ that of $n\times Ac$ or ABCBC, &c. $-\overset{\prime}{v}=$ that of $\overline{n-1}\times Aa$, $-\overset{\prime\prime}{v}=$ that of $\overline{n-1}\times Ab$, &c. we have $\overset{\prime}{t}=\overset{\prime\prime}{t}=\overset{\prime\prime\prime}{t}$, &c. $=Ok=b$, and $\overset{\prime}{v}=-BC-\frac{bc}{2}$, $\overset{\prime\prime}{v}=-BD-\frac{bd}{2}$, &c. and therefore $-\frac{x}{2}^{n-1}\times\overline{\frac{2-cx}{1-cx+xx}+\frac{2-dx}{1-ax+xx}}$, &c. $=\frac{bx-BC-\frac{1}{2}bc}{1-cx+xx}+\frac{bx-BD-\frac{1}{2}bd}{1-dx+xx}$ &c. or $x^{n-1}\times\overline{\frac{2-cx}{1-cx+xx}+\frac{2-dx}{1-dx+xx}+\frac{2-ex}{1-ex+xx}}$, &c. $=\frac{bc+2BC-2bx}{1-cx+xx}+\frac{bd+2BD-2bx}{1-dx+xx}$, &c. which being ſubſtituted, inſtead thereof in $x^{n-1}\times\overline{\frac{2-cx}{1-cx+xx}}$, &c. $=\frac{2nB^2x^{n-1}}{1-2bx^n+x^{2n}}$ (as abovefound) and the whole divided by $2B$, we ſhall have $\frac{C}{1-cx+xx}+\frac{D}{1-dx+xx}+\frac{E}{1-ex+xx}+\frac{F}{1-fx+xx}$, &c. $=\frac{nBx^{n-1}}{1-2bx^n+x^{2n}}$, and, conſequently $\frac{nBx^{n+m-1}}{1-2bx^n+x^{n2}}=\frac{Cx^m}{1-cx+xx}+\frac{Dx^m}{1-dx+xx}+\frac{Ex^m}{1-ex+xx}+\frac{Fx^m}{1-fx+xx}$,&c. Let now $\overset{\prime}{A}x^{m-2}+\overset{\prime}{B}x^{m-3}+\overset{\prime}{C}x^{m-4}$: .... $\overset{\prime}{l}x+\overset{\prime}{s}+\frac{\overset{\prime}{t}x+\overset{\prime}{v}}{xx-cx+1}$ be aſſumed $=\frac{Cx^m}{xx-cx+1}$; then, by Reduction, &c. we ſhall get $\overset{\prime}{A}=\overset{\prime}{C}$, $\overset{\prime}{B}=c\overset{\prime}{A}$, $\overset{\prime}{C}=c\overset{\prime}{B}-\overset{\prime}{A}$, $D=c\overset{\prime}{C}-\overset{\prime}{B}$, &c. &c. where it appears, from the aforenamed Property of the Circle, that $\overset{\prime}{A}$, $\overset{\prime}{B}$, $\overset{\prime}{C}$, ... $\overset{\prime}{t}$, $-\overset{\prime}{v}$, are the Sines of Aa, $2Aa$, $3Aa$, ... $m\times Aa$, $\overline{m-1}$

A *a* reſpectively. Hence, if $\overset{\prime\prime}{A}x^{m-2}+\overset{\prime\prime}{B}x^{m-3}+\overset{\prime\prime}{C}x^{m-4}$ ..... $\overset{\prime\prime}{l}x+\overset{\prime\prime}{s}+\frac{\overset{\prime\prime}{t}x+\overset{\prime\prime}{v}}{xx-xd+1}$ be put $=\frac{Dx^{m}}{xx-dx+1}$, it is manifeſt, that $\overset{\prime\prime}{A}$, $\overset{\prime\prime}{B}$, $\overset{\prime\prime}{C}$ .... $\acute{t}$, $-\acute{v}$ will be the Sines of A *b*, 2 A *b*, 3 A *b* .... $m\times Ab$ and $\overline{m-1}\times Ab$ reſpectively, *&c.* *&c.* Therefore, as it is evident from the above Reaſons, that $\acute{A}+\overset{\prime\prime}{A}$, *&c.* and $\acute{B}+\overset{\prime\prime}{B}$, *&c.* and $\acute{C}+\overset{\prime\prime}{C}$, *&c.* *&c.* muſt all be equal to Nothing, we have $\frac{nBx^{n+m-1}}{1-2bx^{n}+2x^{n}}=$ $\frac{x\times\text{Sine of }m\times Aa-\text{Sine of }\overline{m-1}\times Aa}{1-cx+xx}+\frac{x\times\text{Sine of }m\times Ab-\text{Sine of }\overline{m-1}\times Ab}{1-dx+xx}$ *&c.* which, becauſe A H is $=m\times Aa$ (by Conſtruction), will become $\frac{nBx^{n+m-1}}{1-2bx^{n}+x^{2n}}=\frac{Hb\times x-\text{Sine }(AH-Aa).}{1-cx+xx}$, *&c.* $=\frac{Hb\times x+am\times Ob-Hb\times Om}{1-cx+xx}$, *&c.* Wherefore, the Fluents of $\frac{x\dot{x}}{1-cx+xx}$ and $\frac{\dot{x}}{1-cx+xx}$ being $(Pa:Oa)+\frac{Om}{am}$ (PaO) and $\frac{1}{am}$ (PaO) reſpectively, that of $\frac{Hb\times x\dot{x}+\overline{am\times Ob-Hb\times Om}\times\dot{x}}{nB\times 1-cx+xx}$, *&c.* *&c.* or its Equal $\frac{x^{n+m-1}\dot{x}}{1-2bx^{n}+x^{2n}}$, muſt conſequently be $=\frac{Hb}{nB}$ $(Pa:Oa)$ $+\frac{Ob}{nB}$ (PaO), *&c.* *&c.* which is the very ſame as was before found.

Having thus far effected what was propoſed, it remains next to lay down a Method for finding the aforeſaid Areas ABC, BFGD, in a more eaſy Manner, by Approximation and infinite Series, when that can be done. In order to this, the foregoing Conſtruction of the Points k, P, and C, being ſtill retained, let CR, RS, ST, TU, *&c.* be each equal to AC, and Rr, Ss, Tt, *&c.* perpendicular to AB. Then will $\frac{nz^{\frac{r}{n}p+p}}{pa^{2p}\times Ck}$ into $\frac{Ck}{r+n} + \frac{Rr\times z^{p}a^{-p}}{r+2n} + \frac{-Ss\times z^{2p}a^{-2p}}{r+3n}$ $\frac{-Tt\times z^{3p}a^{-3p}}{r+4n} + \frac{Uv\times z^{4p}a^{-4p}}{r+5n}$, *&c. ad infinitum*, or $\frac{nz^{\frac{r}{n}p-p}}{p\times Ck}$ into $\frac{Ck}{r-n} + \frac{Rr\times z^{-p}a^{p}}{r-2n} + \frac{-Ss\times z^{-2p}a^{2p}}{r-3n}$, *&c.* be $=$ ABCA, or the Area of the Curve whoſe Abſciſſa is z, and Ordinate $\frac{z^{p+\frac{r}{n}\times p-1}}{a^{2p}\pm fa^{p-1}z^{p}+z^{2p}}$; and $\frac{nz^{p-\frac{r}{n}p}}{pa^{2p}\times Ck}$ into $\frac{Ck}{r-n} + \frac{Rr\times z^{p}a^{-p}}{r-2n} + \frac{-Ssz^{2p}a^{-2p}}{r-3n} + \frac{-Tt\times z^{3p}a^{-3p}}{r-4n}$ *&c.* or $\frac{nz^{-p-\frac{r}{n}p}}{p\times Ck}$ *in* $\frac{Ck}{r+n} + \frac{Rr\times z^{-p}a^{p}}{r+2n} + \frac{-Ss\times z^{-2p}a^{2p}}{r+3n}$ $\frac{-Tt\times z^{-3p}a^{3p}}{r+4n}$, *&c.* $=$ BFGD, or the Area of the Curve whoſe Abſciſſa is z, and Ordinate $\frac{z^{p-\frac{r}{n}p-1}}{a^{2p}\pm fa^{p-1}z^{p}+z^{2p}}$. The Reaſons whereof, from the former Part of the foregoing Solution, will appear manifeſt.

FINIS.

E R R A T A.

PAGE 5. l. 17. for *q.r* read *g.r*, p. 28. l. 14. r. $\frac{vv}{2} = \frac{2rx^{n+1}}{\overline{n+1} \times a^{n}}$, l. 18. for a^2v^2 r. x^2v^2, l. laſt, for $\sqrt{m^2, \&c.}$ r. $\frac{\dot{z}}{\sqrt{m^2, \&c.}}$, p. 31. l. 5. for *the Ratio*, r. *which is to Unity in the Ratio*, p. 32. l. 18. for *than*, r. *than as*, p. 42. l. 24. for R*s* r. *ks*, l. 25. for *vr* r. *sr*, p. 48. l. 25. for S C r. *l* C, p. 49. l. 16. for 5 *or* 6 r. 7 *or* 8, p. 51. l. 19. for $52'\ 50''$ r. $51'\ 52''$. l. penult. for $52'\ 51''$ r. $51'\ 52''$, p. 53. l. 20. for $\frac{b}{g}$ r. $\frac{v}{g}$, p. 57. l. 24. for *Specifick Gravity* r. *Specifick Gravity in Air*, p. 61 l. 20. for $\frac{y^2}{2p}$ r. $\frac{y^2}{p}$, p. 67. l. 18. for *of* r. *to*, p. 76. l. 12. for *of n* r. *n*, p. 85. l. 10. for $1 \frac{x}{y}$ r. $\frac{x}{y}$, p. 93. l penult. for *Allowances* r. *Allowance*, p. 101. l. laſt, for 2, 3, *&c.* r. 1, 2, 3, *&c.* p. 103. l. 2. for $x=0$ r. $x=100$, l. 3. for *Cor.* II. r. *Cor.* III. p. 104. l. 24. for nx^{n+1} r. x^{n+1}, l. penult. for dx^{n} r. dz^{n}, p. 105. l. 2. for dzn r. dz^{n}, l. 15. for *be* r. *are*, and for *are* r. *be*, l. 18 for A H r. A R, p. 113. l. 24. for $\overline{1 \backsim x}|^{2} v$ r. $\overline{1 \backsim x}|^{2} = v$, p. 121. l. laſt, for x^{n-r} r. x^{n+r}.

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Auctore GULIELMO BRACKENRIDGE,

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